A MANUAL OF HYDRAULICS

BY

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PREFACE.

THE growing needs of modern industry have given rise to a corresponding development in energy producers.

In recent years steam engines have been brought to such a state of perfection, that it appears impossible for them to be much further improved in the future. Again, the Parsons, Curtis, and Laval steam turbines have each contributed their share to the production of mechanical energy. Also explosion engines, using either oil vapour, gas, petroleum, or alcohol, have made possible the transformation of heat into mechanical work, with an efficiency which, until recently, was quite unlooked for.

But all these engines consume great quantities of coal, that "black bread" of industry, whose price, increasing with its use, causes the loss of all the fruits of economies attained by successive improvements in the efficiency of heat engines.

It is because industry has sought other sources of energy in addition to the fuel which lies underground, that water-power has been exploited, and that use has been made of that "white oil" which collects on the mountains and is conveyed in rivers to work hydraulic machinery.

Of course the utilisation of waterfalls is not of recent date, but their applications have been very limited in the past because the energy obtained could not be distributed in the great centres where it was needed.

Since, however, electric currents have rendered possible the transport of water forces stored up in mountainous districts to popular centres, there has been a veritable revolution.

From that time onwards hydraulic power installations have increased enormously, for it has become possible to

put to profitable use reservoirs of incalculable energy, thanks to the marvellous progress of electrical science.

The Alps and the Pyrenees offer for our use almost inexhaustible stores of water-power, not to mention the huge American installations like that at Niagara, where already more than one hundred thousand horse-power is being used.

There is therefore a vast field for exploitation by engineers, and it may be said that the twentieth century should be called "the age of water-power," just as the last century has beeu called "the steam age."

Thus problems concerning hydraulic power are more and more the order of the day; may we be allowed to suggest, therefore, that this *Manual of Hydraulics* supplies an industrial need, and comes at exactly the right time.

In writing this work we were inspired by the same ideas, and have followed the same plan as guided us in the production of our *Traité d'Électricité Industrielle.*

The book itself is not a purely descriptive work designed merely for popular use, nor is it an abstruse treatise suitable only for engineers versed in higher mathematics. It is a text-book of applied hydraulics, in which complete technical theories and all useful calculations for the erection of hydraulic plant are presented. In it no recourse is made to other operations than those of arithmetic and elementary geometry.

It has been our aim to make the science of the hydraulic power industry available for all engineers, architects, and contractors, who may be called upon to study and carry out installations of this description.

We trust that this text-book will render real service, and that we have produced, though not on an elaborate scale, a work which will be both useful to and appreciated by all those interested in hydraulic problems.

R. BUSQUET.

LYONS, *March*, 1905.

TRANSLATOR'S PREFACE.

THIS book expounds the principles underlying the use of water-power and discusses the application of these principles to almost every type of hydraulic prime mover in commercial use, shewing the relative merits of and the circumstances favourable to each type.

In translating, the same simple arithmetical methods used in the original have been adhered to, so that only a very elementary knowledge of arithmetic and geometry is necessary in order that the whole of the numerous examples may be followed. All dimensions have been changed into ordinary British units, and the constants occurring in the various formulae have of course been modified to suit the change.

In several instances slight modifications or additions have been introduced by the translator, and such changes are indicated in foot-notes, except in cases of obvious inaccuracy.

It is perhaps advisable to explain that, wherever the term "inclination of an angle" is used, it is the tangent of the angle that is meant, so that the slope of one line with respect to another is expressed by the increase of perpendicular distance between the two lines per unit length of that one to which the perpendicular is drawn.

A. H. PEAKE.

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CHAPTER I.

FUNDAMENTAL LAWS.

1. Pressure.—Work. As everyone knows, water has great fluidity, that is to say, the various molecules of which a definite mass of water is composed move past each other very easily; but this fluidity is not perfect, and the relative movements of the molecules or of the stream-lines cause friction both between the different parts of the liquid itself, and also between the liquid and the surrounding solid bodies.

Water may be at rest or in motion, and therefore the subject divides itself naturally into Hydrostatics, which treats of liquids at rest, and Hydrodynamics, which deals with the study of liquids in motion.

The principal laws of Hydrostatics are well known, especially those concerning the equilibrium of fluids in communicating vessels, and the transmission of equal pressure in all directions in a liquid at rest. We will draw attention to one definition only, the important one known as *free surface level.* Consider the case of a pipe containing water either flowing or at rest; this water, according to the principles above mentioned, exerts a pressure on the inside surfaces of the pipe in all directions, including the upward direction. This pressure depends upon the difference in vertical height between the point considered and the surface of the water.

In order that movement may be produced, there must be a difference of level, causing a corresponding difference of pressure, between the source and end of the movement, in proportion to the flow produced. Under these circumstances, whether the difference of level is provided by simply inclining

the tube to the horizontal, or whether it is due to a reservoir lying higher than the pipe, the water will be set in motion and may be discharged through an orifice.

If the pipe be closed at its lower end, and a small hole be made at a point in the upper surface, a *jet of water* will be obtained which will spurt to a height which will be the greater according to the difference of level between this point and the upper end of the pipe; there will be then a discharge from the pipe and a flow inside it; friction of the stream-lines against one another and against the inner faces of the tube will determine the resistance to the flow of the liquid, and will tend to reduce the vertical height of the jet.

If instead of a discharge orifice, a tube be erected, of sufficient length to reach considerably beyond the height of the jet in the preceding case, so that no discharge of water can take place, the column of water in the tube will rise to exactly the same height as the surface of the water in the upper end of the pipe, or in the reservoir; in other words the reservoir and the tube form two branches of a system of communicating vessels and the surfaces of the water will be at the same height in the two branches (fig. 1).

Let us suppose for example that the tube has an inside cross-sectional area of 1 square inch; this will also be the section of the column of water. Let us consider this column of water prolonged below the end of the vertical tube to the centre G of the pipe. At this point, the pressure exerted by the column in virtue of its weight, will be equal to the volume of the column of water multiplied by its density or weight per cubic inch. This will of course be different for different liquids, and we will designate it by the letter d.

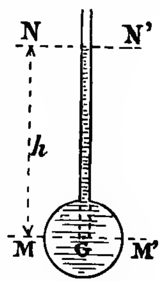

Fig. 1.
Pressure column.

The volume being the product of the section and the height h, and the section being taken equal to 1 square inch by hypothesis, we have:

$$\text{Weight of the column} = d \times h \times 1 = d \times h.$$

It must be observed that this weight does not constitute the whole pressure borne by the area which serves as the base of the column. The column itself may be regarded as being surmounted by a column of air of the same section, whose weight should be added to that of the column of water.

Thus if we express the pressure per square inch of horizontal section exerted at the centre of the pipe by p, and if P_a be used to denote the atmospheric pressure on the top of the column, a pressure which is practically the same at the level of the point considered, we may write :

$$\text{Total pressure } p = d \times h + P_a,$$

or, subtracting P_a from each side of the equation, which in no way alters the equality :

$$d \times h = p - P_a.$$

h represents what is called the *head* at the point of the pipe considered, and the level NN′, at which the top of the column remains stationary, is called the *free surface level* of the same point in the pipe.

Thus far, we have supposed that the pressure tube was the only opening in the main pipe, and therefore that the liquid was in a state of rest throughout.

If, however, a continuous flow is produced by another orifice, so that the whole of the fluid in the main pipe is in motion, the level in the pressure tube will be lowered, even though the upper level in the reservoir is kept constant, and the height of the column will be reduced in proportion to the flow through the pipe.

If now the main pipe be provided with pressure tubes spaced at equal intervals throughout its length, it will be found that the level falls progressively in these tubes showing loss of pressure in consequence of the frictional resistance to flow.

The different levels attained in the various tubes are the free surface levels corresponding to the pressures in the different parts of the pipe.

These successive losses of pressure or *losses of head* are a

consequence of the laws of energy. Water raised to a definite height or stored in a conduit, inclined in such a manner as to have a certain difference of level between its extremities, is a reservoir of mechanical energy.

Every lb. of water in falling through a height of 1 foot does a certain amount of work, to which the name *foot-lb.* has been given, and which has been chosen as the unit of energy.

Now, every lb. of water in the reservoir before falling possesses latent or potential energy to the extent of 1 foot-lb. for every foot of elevation above any given level, and in consequence a certain pressure exists at the lower end of the pipe, which may be measured by the height of the free surface level at that point. When, however, the water flows, a certain amount of the stored energy is used up in doing work against friction, so that the pressure at the lower end of the pipe is no longer that corresponding to the height of the reservoir, but corresponds to the height of an imaginary reservoir which is lower than the former by an amount proportional to the work done in overcoming friction.

Thus the reduction in pressure or head between the states of rest and motion is produced because work is done.

Bearing these principles in mind, let us pass on to consider some problems in hydrodynamics.

2. Bernoulli's theorem. A theorem which is very often made use of in hydrodynamics is one formulated by Daniel Bernoulli.

In order to understand the object of this fundamental theorem, let us consider water flowing under the action of gravity, from a level H to a point of lower level B in a tube of any shape whatever, whether straight or curved, parallel or tapering (fig. 2).

Having settled on a particular arrangement, let us in the first place consider what is the velocity of the water at different sections of the tube such as AB, *ab*, etc., and secondly what are the pressures at these same sections.

Bernoulli's theorem is for this purpose, and is founded on

the general laws of gravity and of falling bodies, either solid or liquid.

When a body is allowed to fall freely under the action of gravity it falls with a uniform acceleration, its speed increasing every second by about 32 feet per second in this latitude.

Thus after 10 seconds the velocity acquired will be 320 feet

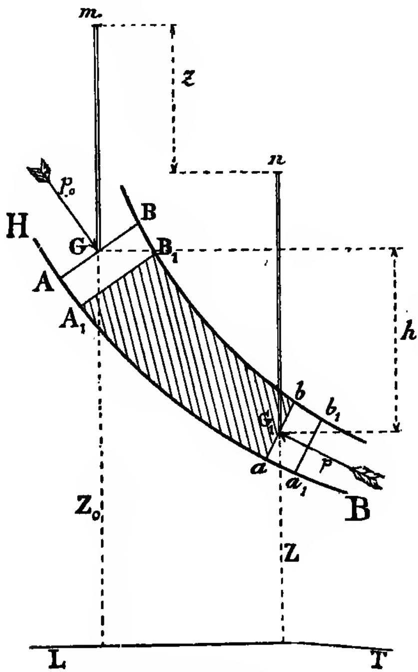

Fig. 2. Bernoulli's theorem.

per second, or at least this would be the velocity if the fall were to take place in a vacuum, so that the resistance of the air did not interfere with the motion.

In the case of a body falling freely from a state of rest, the speed is at first nothing and increases uniformly under the action of gravity until at the end of the first second it is

32 feet per second. Therefore it will not have travelled through 32 feet in the first second of its motion from rest, but only through a distance corresponding to its average velocity, namely:

$$\frac{0+32}{2}=16 \text{ feet.}$$

The number 32 is generally denoted by the letter g, which simply represents the acceleration due to the action of gravity.

Since both the velocity and the drop or space travelled through increase with time, there will be a definite velocity corresponding to each particular distance dropped, such that if V and H be used to represent at any particular point the velocity and the distance dropped respectively, it will be found that by directly applying the law of gravity just mentioned, the following relation holds:

$$V=\sqrt{2gH}.$$

Suppose a body has fallen 81 feet from rest, its velocity at that point will be from the above formula:

$$V=\sqrt{2\times 32\times 81}=72 \text{ feet per second.}$$

On squaring each side of the equation, which of course does not affect the condition of equality, we obtain:

$$V^2=2gH,$$

and dividing each side by $2g$:

$$\frac{V^2}{2g}=H.$$

By means of this latter formula, if V be known, the corresponding drop H can be deduced; thus in the example chosen, $V=72$, $2g=64$, hence:

$$H=\frac{5184}{64}=81 \text{ feet.}$$

The tube under consideration being of irregular form, has different sectional areas at different points. For instance, the area S_0 of the cross-section AB is greater than the cross-section S at *ab*. It should be obvious that the quantity of water

passing the first section in any interval of time is equal to the amount which passes the second section in the same interval, therefore also the speed of flow for the same volume of water will be greater where the sections are smaller, or in other words, the velocity will be inversely proportional to the area of cross-section.

Let q be the volume of water which passes every cross-section per second; also let S_0 and S be the areas of the sections at AB and ab, and let V_0 and V be the corresponding velocities.

The volume passing per second may be regarded as a cylinder of liquid, having for base the cross-section of the stream, and for length the distance travelled through per second, or the velocity, at that section. Therefore:

$$\text{Volume} = q = S_0 \times V_0 = S \times V.$$

Considering the pressures exerted on the body of water between the two sections AB and ab, the first acts from left to right assisting the flow, but the second acts from right to left against the direction of the flow. Thus the water contained between these two sections is acted upon by a force $p_0 \times S_0$ on one side, and a force $p \times S$ on the other, since of course the pressures per square inch p_0 and p must be multiplied by the number of square inches in each section.

But this quantity of water is again acted upon in a vertical direction by gravity, with a force equal to its own weight.

The motion of the liquid is due then to the combined action of these different forces.

Now whenever a mass is moved by the action of forces, work is done in proportion to the displacement produced, and this work is equal to the sum of the products of each force and the distance moved through in the direction in which the force acts.

Let us see then what is the work done by each force in one second.

The distance travelled through per second is simply the velocity.

The work done by the force $p_0 \times S_0$ is therefore :

$$p_0 \times S_0 \times V_0 = p_0 \times q,$$

since :
$$S_0 \times V_0 = q.$$

Similarly the work done against the force $p \times S$ is :

$$p \times S \times V = p \times q.$$

This latter is work given out by the water under consideration, on the water in front of it.

Now in order to obtain the work done by the weight of the mass AB, *ab*, suppose that the layer of liquid AB has moved to A_1B_1 at the end of one second; the volume which has passed during that time will be ABA_1B_1 and this volume will be equal to

$$q = S_0 \times V_0.$$

It is clear that if during the same time the lower slice *ab* has moved to a_1b_1, the volume aba_1b_1 will be exactly equal to ABA_1B_1, for this is also equal to q.

The result then is the same as if the intermediate mass A_1B_1ab had remained stationary, while the prism of water ABA_1B_1 had been transported to aba_1b_1 effecting in this movement a vertical drop equal to the difference of levels between the centres of gravity of the two sections considered.

Hence the work of gravity is done by a mass ABA_1B_1 of weight equal to the volume multiplied by the density, $q \times d$ say, which drops through a height h :

$$\text{Work} = q \times d \times h.$$

The total resultant work put into the column of water in the fall is the sum of the work done by each separate force; and work given out is of course reckoned as negative work put in.

We have then :

$$\text{Total work} = p_0 \times q \;-\; p \times q \;+\; q \times d \times h.$$

To obtain from this the expression of Bernoulli's theorem, we must bring in a mechanical principle which follows from the law of the conservation of energy.

This law states that the work done by the different forces is not lost, but reappears in another form. In fact the mass of water itself stores up the energy developed by the acting forces; a velocity which depends on the work stored has been given to it, and if, for example, this water is delivered to a water-wheel, it may transmit the energy it possesses to the wheel and lose its velocity. Therefore it is in the form of motion that the energy or work is stored.

Actually the power stored in the moving mass is proportional to its mass and to the square of the velocity which it possesses.

In mechanics the mass of a body is expressed by the weight divided by the gravitational constant, namely $\frac{P}{g}$, so that the work stored is expressed by:

$$\frac{1}{2} \times \frac{P}{g} \times V^2.$$

The factor $\frac{1}{2}$ follows from considerations already dealt with.

It follows that the energy given to, or the work stored in, a mass whose velocity is increased from V_0 to V is:

$$\frac{P}{2g} \times (V^2 - V_0^2),$$

or, in the case we are considering, P being equal to $q \times d$:

$$\frac{q \times d}{2g} \times (V^2 - V_0^2).$$

We have now only to apply the law of the conservation of energy, and say that the total work done on the water by the separate forces is equal to the energy stored in the water in virtue of its increased velocity, that is:

$$p_0 \times q - p \times q + q \times d \times h = \frac{q \times d}{2g} \times (V^2 - V_0^2).$$

But each side of this equation can be divided by $q \times d$, therefore:

$$\frac{p_0}{d} - \frac{p}{d} + h = \frac{V^2}{2g} - \frac{V_0^2}{2g}.$$

If we put: $\frac{p_0}{d} = h_0$ and $\frac{p}{d} = h_1$,

where h_0 and h_1 are the heights to which the liquid at G and G_1 would rise if small vertical tubes were erected at these points, for:

$$p_0 = d \times h_0 \text{ and } p = d \times h,$$

we then have: $$h_0 - h_1 + h = \frac{V^2}{2g} - \frac{V_0^2}{2g}.$$

In the figure Gm represents h_0 and $G_1 n$ represents h_1, and we see that $h_0 - h_1 + h = z$; z being the difference in height of the free surface levels.

Finally therefore:

$$\frac{V^2}{2g} - \frac{V_0^2}{2g} = z.$$

This is the expression of Bernoulli's theorem.

In this expression, $\frac{V^2}{2g}$ and $\frac{V_0^2}{2g}$ are equivalent to definite heights as has been already explained, and the difference between them is the drop that would be necessary for water falling freely to have its speed increased from V_0 to V.

Bernoulli's theorem may therefore be expressed thus: **When between two consecutive sections of a current of liquid, an increase of velocity is produced, the imaginary fall that would be necessary to produce such a change in the velocity is equal to the difference in the free surface levels at the particular sections considered.**

Therefore we may say that the difference in height of the free surface levels, or the loss of head, causes the increase of velocity.

So far, it has been assumed that the friction between the liquid molecules themselves, and against the sides of the tube, is negligible. If this is not so, the increase in speed will be less than that corresponding to the height z, for a part of the potential energy will be used in overcoming friction, and the height available for acceleration will be reduced by this

amount. In this way the whole potential energy may be used in overcoming friction, so that the velocity, instead of being increased, remains uniform.

Referring to the figure, we can now put Bernoulli's expression in another form which possesses certain practical advantages.

Suppose Z and Z_0 are the heights of the centres of gravity of the two sections we are considering above some given level LT, we have from the figure, $h = z_0 - z$.

Therefore the formula may be written :

$$\frac{V^2}{2g} - \frac{V_0^2}{2g} = \frac{p_0}{d} - \frac{p}{d} + (Z_0 - Z),$$

and changing the terms from one side of the equation to the other by adding thereto or subtracting them from each side simultaneously :

$$\frac{V^2}{2g} + \frac{p}{d} + Z = \frac{V_0^2}{2g} + \frac{p_0}{d} + Z_0.$$

The separate terms in each member of this expression represent heights : $\frac{V^2}{2g}$ is the height corresponding to the velocity at the section *ab*, $\frac{p}{d}$ is the head, or the pressure measured in terms of the height of a column of liquid at the same section, Z is the height of its centre of gravity above the plane of comparison, and the other terms apply in the same way to the other cross-section.

Thus the sum of the heights as defined above, corresponding to a section such as AB, is equal to the sum of the similar heights for any other section *ab*, so that Bernoulli's theorem may again be expressed as follows :

The sum of the velocity head, the pressure head and the altitude is the same for all sections.

3. Flow through sharp-edged orifice. We can now make use of Bernoulli's theorem to solve a number of problems of practical interest.

The first is the determination of the velocity and quantity of water discharged through a sharp-edged orifice.

In hydraulics an orifice is said to have sharp edges whatever be the nature of the material in which it is made, provided that the thickness of the edges is less than half the least diameter of the orifice.

If the orifice itself is small, the edges should be very sharp, and the liquid will then flow past them without appreciable friction, as if they had no thickness at all (fig. 3).

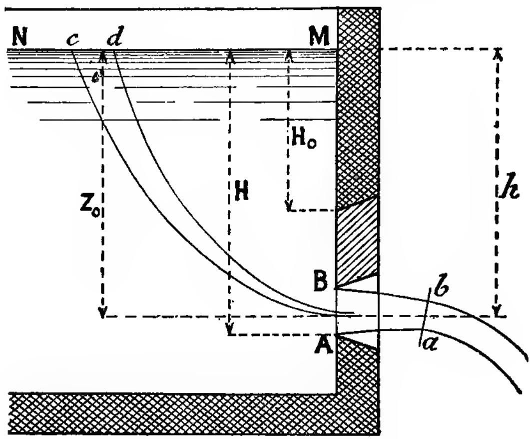

Fig. 3. Sharp-edged orifice.

When such an orifice is made in the side of a tank, it is found that the issuing jet of liquid contracts and has a minimum section *ab* at a certain distance from the orifice, that is to say if S be the area of the orifice AB, the jet will assume a conical form whose least section *ab* will be only a fraction of S.

From among the stream-lines or tubes which go to supply the jet, we may consider the tube which is continued to the surface NM at *cd*, and apply Bernoulli's theorem to it.

Let us take for our plane of comparison, or datum level, the horizontal plane which passes through the centre of

gravity of the contracted section *ab*. Let $\frac{V_0^2}{2g}$, $\frac{p_0}{d}$ and Z_0 be the velocity head, pressure head, and altitude at the section *cd*, and $\frac{V^2}{2g}$, $\frac{p}{d}$ and Z the respective corresponding quantities at the section *ab*. Then:

$$\frac{V^2}{2g} + \frac{p}{d} + Z = \frac{V_0^2}{2g} + \frac{p_0}{d} + Z_0.$$

But the velocity of the molecules at *cd* is very feeble, if we suppose that the surface of the reservoir is large compared with the size of the orifice, and we may therefore neglect it, and put $V_0 = 0$.

Further, p_0 the pressure at the surface of the reservoir, is the same as that at *ab* in the jet, and is nothing but the pressure of the atmosphere; in fact, it is found that the elementary stream-lines composing the jet of liquid describe the same parabolic trajectory as if they were isolated and independent; the stream-lines inside the jet are therefore subject to the same conditions as those forming the outside layers, and the atmospheric pressure which is exerted on the outer stream-lines consequently acts to the same extent on the others throughout the whole section *ab*.

Finally $Z = 0$, since it was chosen to be so, and $Z_0 = h$.

Under these conditions the formula becomes:

$$\frac{V^2}{2g} + \frac{p}{d} + 0 = 0 + \frac{p_0}{d} + h,$$

or:

$$\frac{V^2}{2g} = \frac{p_0}{d} - \frac{p}{d} + h.$$

And if, as we have supposed, the liquid flows into the air in such a manner that $p_0 = p = p_a$, where p_a denotes the atmospheric pressure, we have simply:

$$\frac{V^2}{2g} = h.$$

So that the velocity of the water corresponds to the height of the surface above the orifice, and is just the same as if the water had fallen freely from this height.

The velocity of the jet at the contracted section *ab* may be obtained by writing:

$$V = \sqrt{2gh}.$$

If for example the height $h = 4$ feet, then:

$$V = \sqrt{2 \times 32 \times 4} \text{ from which } V = 16 \text{ feet per second.}$$

The discharge, or the volume flowing per second, is got by multiplying the area of section by the velocity at that section.

Now it is impossible to determine from theoretical considerations the ratio of the contracted section to the area of the sharp-edged orifice, but it has been found by experiment that when the inner face of the reservoir is plane, this ratio, which is called the coefficient of contraction, never differs much from the value 0·62.

Hence we have the relation:

$$s = 0{\cdot}62S,$$

s being the area of the contracted section.

The volume discharged per second is therefore:

$$q = 0{\cdot}62 \times S \times V.$$

So that if the orifice has an area of 1 square inch $= \frac{1}{144}$ square feet, the delivery will be $0{\cdot}62 \times 16 \div 144$ $= 0{\cdot}07$ cubic feet per second approximately.

If the delivery is to remain uniform, it is of course essential that the level in the reservoir must not be changed, so that there must be a steady flow into the reservoir to compensate for the discharge through the orifice.

If instead of an orifice of small dimensions such as we have been considering, we have a large rectangular opening as more often occurs in practice, we may suppose the opening to be divided up into a number of elementary areas each very small in height, but which altogether make up the whole section.

Each elementary section will give a discharge depending on its depth below the level NM, the quantity being actually proportional to the square root of this depth, as has just been shewn.

Therefore the total flow through the whole orifice will be proportional to the mean square root of the depth. Thus, if all the elementary areas be supposed of equal size, it would be necessary first of all to obtain the depth of the centre of each elementary area, then to calculate the square root of each of these depths, and lastly to take the average of these square roots by adding them together and dividing the sum by the number of elementary areas into which the whole area is divided.

The value of the mean square root of the depth thus obtained is a measure of the mean velocity of the liquid in the orifice. However*, although the mean square root of the depth and the square root of the mean depth are not identical terms, yet as a rule the values of the two are not very different, and in most cases it will suffice to calculate the latter, namely the square root of the mean depth, which will give the resulting velocity to a close degree of approximation.

Therefore if H_0 and H are the depths of the horizontal edges of the orifice below the level in the reservoir, we may write:

$$q = 0{\cdot}62S \times \sqrt{2g \cdot \frac{H_0 + H}{2}}.$$

That is to say, the flow is approximately obtained, just as in the preceding case, by multiplying the area of the contracted section by the velocity resulting from a free fall from the surface level in the reservoir to the centre of the orifice.

And the smaller the vertical dimension of the orifice in proportion to the depth below the surface, the more accurate will be the result.

Moreover the coefficient of contraction is not absolutely constant. It increases as the vertical dimension of the orifice diminishes; it also varies with the head.

It really varies only between 0·62 and 0·64 and therefore the figure 0·62 is sufficiently accurate, for since friction has

* This sentence, and the preceding explanation that the flow is proportional to the mean square root of the depth, have been added by the translator.

been neglected, the flow is a little less than that indicated by the theory.

We see then, that to obtain the discharge through a sharp-edged rectangular orifice, the head is taken as the height of the water level above the centre of the orifice; but evidently the reasoning employed in using Bernoulli's theorem is independent of the shape of the orifice, and is quite applicable to a square, rectangular or circular orifice. So that in every case if we use h to denote the depth of the centre of the orifice below the level of the water, the volume discharged per second will be given by the general formula:

$$q = 0{\cdot}62S \times \sqrt{2gh}.$$

4. Drowned orifice with sharp edges. In the above example we have supposed that the water was discharged freely into the atmosphere; it is also interesting to examine the case in which a liquid flows from one reservoir to another through a sharp-edged orifice made in the partition separating the two reservoirs, the level of the water in the lower reservoir being maintained at a constant height above the orifice.

The orifice in this case is said to be *drowned.*

Let us imagine that a vessel is attached to the right of the reservoir already considered, and that the water flows from the latter into this, while the level is maintained in it at a distance H_0 say, below the level NM (fig. 3).

Let us again choose as datum level the plane passing through the centre of the orifice, and as before let $\frac{V_0^2}{2g}$, $\frac{p_0}{d}$ and Z_0 be the velocity head, pressure head and altitude at the upper section cd.

Further, let $\frac{V^2}{2g}$, $\frac{p}{d}$ and Z represent respectively the same quantities at the orifice. Then by Bernoulli's theorem:

$$\frac{V_0^2}{2g} + \frac{p_0}{d} + Z_0 = \frac{V^2}{2g} + \frac{p}{d} + Z.$$

Now $Z = 0$ by hypothesis, and the pressure p is made up of

two parts, the atmospheric pressure p_a and that due to a column of liquid of height $(Z_0 - H_0)$. The pressure corresponding to this height is $(Z_0 - H_0)\,d$ as already shown, hence:

$$p = p_a + (Z_0 - H_0)\,d,$$

or:

$$\frac{p}{d} = \frac{p_a}{d} + Z_0 - H_0,$$

the formula therefore simplifies to:

$$\frac{V_0^2}{2g} + \frac{p_0}{d} + Z_0 = \frac{V^2}{2g} + \frac{p_a}{d} + Z_0 - H_0.$$

But $V_0 = 0$ and p_0 is simply the atmospheric pressure p_a, so that finally:

$$\frac{V^2}{2g} = H_0, \quad \text{or } V = \sqrt{2gH_0}.$$

Hence we see that the effective head is the difference of the levels in the two reservoirs.

It should be noticed further, that the velocity of discharge is independent of the depth of the orifice below the level in the right-hand vessel, so that whether the orifice is situated near the bottom of the reservoir, or whether its upper edge is only just below the level in the vessel into which the liquid flows, the velocity is always the same.

5. Orifice prolonged by open channel. A third case which often occurs in practice is that in which a channel or open trough is attached to the orifice so that the stream-lines are guided for a certain distance on emerging.

For this purpose it is necessary that the channel should possess a definite inclination such that the stream-lines maintain the speed which they possess at the contracted section.

This problem can be solved by the reasoning preceding, for the channel may be regarded as an exterior reservoir filled by the stream of water.

The head being given as before by the difference of levels in the main reservoir and in the channel, namely, the vertical

distance h say between NM and the free surface level of the stream ; the expression for the velocity will always be :

$$V = \sqrt{2gh}.$$

6. Various values of contraction coefficient. Whether the water escapes freely into the atmosphere, or whether it flows from one vessel to another through a drowned orifice, or again, whether it flows through a nozzle or an open channel, in every case the stream is contracted at a certain distance from the orifice.

It is possible to reduce this contraction to some extent by arranging the sides of the vessel near the orifice so as to guide the stream-lines in a nearly parallel direction ; in this way the coefficient of contraction is increased. It is very nearly unity for orifices with well-rounded edges, and may reach the value 0·75 for small heads of water and for orifices whose diameters are less than $\frac{3}{4}$ of an inch.

In the case of complete contraction, when there are no directing surfaces, the coefficient of contraction, namely, the ratio between the smallest section and the area of the orifice is 0·62 on the average, as already stated.

Various investigators have found that for a rectangular orifice 0·4 inches square, the distance of the contracted section from the orifice varies between 0·16 and 0·2 inches.

There are certain books which contain tables of experimental results, and which give the coefficients applicable to all heads and sizes of orifice ; but in the majority of cases the figures given will be sufficient for practical purposes.

7. Flow over a dam or weir. Let us now consider the case of water flowing over a weir (fig. 4).

Here the water is in contact with the horizontal crest of the weir, and with the vertical sides which limit the length of the weir.

The level in the reservoir is necessarily higher than the weir. When a flow of water occurs under such conditions, it is found that at a little distance from the weir, at a point m

for example, the surface of the water curves downwards and falls more and more until the weir is reached, so that the sheet of water above the plane *nab* has a minimum thickness at the weir.

This case is exactly the same as that already examined, in which the flow took place through an orifice prolonged by an open channel.

In fact it may be imagined that a vertical board is placed at P*k* with its lower edge at a height *ka* above the crest of the dam, equal to the thickness of the sheet of water, and the sill of the weir may be regarded as an open channel.

In order that the same reasoning may apply, the thickness of the dam wall must be sufficient for all the stream-lines to take a horizontal direction in passing over the crest after leaving the section *ka*. It may then be assumed that the various stream-lines move freely as if they were isolated and merely subjected to the hydrostatic pressure.

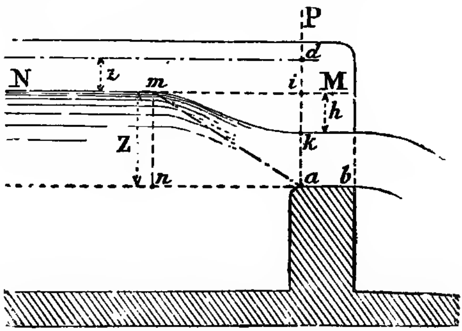

Fig. 4. Dam or weir.

If we suppose also that the level MN is maintained by a continuous supply of water to the reservoir, and that the velocity of the water at the section *nm* is negligible in comparison with that at *ka*, we obtain at once, by comparison with the preceding cases, the formula :

$$V = \sqrt{2gh},$$

in which V is the velocity of flow on the crest of the dam, and h is the difference in vertical height between the main level in the reservoir and the surface of the sheet of water on top of the dam.

If the value of h be known, V can easily be calculated, and the discharge Q may be deduced as usual by multiplying this velocity by the section of the sheet of water.

Let L be the length of the weir between the vertical sides, the thickness of the current sheet on the sill is $(Z - h)$, so that the section of the stream is $L \times (Z - h)$, from which the discharge will be :

$$Q = L \times (Z - h) \times \sqrt{2gh}.$$

It is necessary then to know h, and this may be determined either by calculation or by experiment.

The possibility of calculating h depends on the fact that it will be such that the flow of water will be the maximum possible under the circumstances.

It is obvious that h will take up some definite value between the value Z on the one hand and zero on the other.

It is impossible for h to be equal to Z, for this means that there is no depth of water flowing over the weir, which is the same thing as saying there is no flow, and the surface of the water would have some such slope as *ma*, which is incompatible with equilibrium ; on the other hand h cannot be zero, for this would correspond to a large flow of water without any loss of head to cause it.

Examination of the formula bears out these facts, for if $h = Z$, or $h = 0$, in either case $Q = 0$, which is absurd.

Since therefore h lies between 0 and Z it will be some particular fraction of Z and may be written equal to $K \times Z$, where K is the fraction required.

Putting this in the formula, we get :

$$Q = L \times (Z - KZ) \times \sqrt{2gKZ},$$

which may be written again thus :

$$Q = LZ\sqrt{2gZ} \times (1 - K)\sqrt{K}.$$

It should be noticed that in the second member of this expression Z is known, and the terms $(1-K) \times \sqrt{K}$ are the only unknown quantities.

The value of these depends solely on the fact that Q is to be a maximum, so that all that is necessary is to determine what value of the fraction K will give the greatest discharge Q.

It is easy to see the value sought must be $\frac{1}{3}$, in which case, putting $K = \frac{1}{3}$ in the above expression, we get:

$$(1 - \tfrac{1}{3}) \times \frac{1}{\sqrt{3}} = 0{\cdot}385.$$

For if $K = \frac{9}{30}$, a value slightly less than $\frac{1}{3}$ or $\frac{10}{30}$, the result is 0·3834, and for $K = \frac{11}{30}$, which is slightly greater than $\frac{1}{3}$, the result is 0·3835, and hence it is for $K = \frac{1}{3}$ that the expression has its maximum value.

The formula for the volume per second finally becomes then:

$$Q = 0{\cdot}385\ LZ\sqrt{2gZ}.$$

This rule is also confirmed by experiment.

It may be that, contrary to supposition, the velocity V_0 at the section *mn* is not negligible as compared with the velocity of flow at the weir V, in which case we must write the complete Bernoulli formula:

$$\frac{V^2}{2g} + \frac{p_0}{d} + Z - h = \frac{V_0^{\,2}}{2g} + \frac{p_0}{d} + Z,$$

which reduces to: $$\frac{V^2}{2g} = \frac{V_0^{\,2}}{2g} + h.$$

Thus the imaginary drop $\frac{V^2}{2g}$ which produces the velocity is the actual fall h increased by the additional amount $\frac{V_0^{\,2}}{2g} = z$, and we may write:

$$\frac{V^2}{2g} = h + z,$$

from which: $$V = \sqrt{2g \times (h + z)}.$$

The result therefore is the same as if there had been still water at *mn* with its surface at a height $Z + z$ above the crest

of the weir, while the fall in level, kd, between the still water surface and the surface of the sheet passing over the dam $=(h+z)$. The expression for Q becomes then:

$$Q = L \times (Z - h) \times \sqrt{2g \times (h+z)},$$

which may be re-written:

$$Q = L \times [(Z+z) - (h+z)] \times \sqrt{2g \times (h+z)}.$$

But by analogy with the former case $(h+z)$ must equal $\frac{1}{3} \times (Z+z)$ and may be replaced by it in the formula; moreover it may be seen that the expression obtained before, applies in this case, if we consider that the height Z has become $(Z+z)$, and we may write at once:

$$Q = 0{\cdot}385\, L \times (Z+z) \times \sqrt{2g \times (Z+z)}.$$

As a numerical exercise, let:

$$V_0 = 6 \text{ ft. per sec.,} \quad Z = 1 \text{ ft.,} \quad \text{and} \quad L = 30 \text{ ft.,}$$

we have in the first place:

$$z = \frac{6^2}{2 \times 32} = 0{\cdot}563 \text{ ft.,}$$

and consequently:

$$Z + z = 1{\cdot}563 \text{ ft.}$$

The discharge is then:

$$Q = 0{\cdot}385 \times 30 \times 1{\cdot}563 \times \sqrt{64 \times 1{\cdot}563} = 180{\cdot}5 \text{ cu. ft. per sec.}$$

The velocity at the weir will be:

$$V = \sqrt{2g \times (h+z)} = \sqrt{2g \times \frac{Z+z}{3}} = 5{\cdot}78 \text{ ft. per second.}$$

The thickness of the sheet of water passing over the weir is:

$$ak = Z - h = \tfrac{2}{3}(Z+z) = 1{\cdot}042 \text{ ft.}$$

so that we may again calculate the discharge Q; for the section of the stream is:

$$S = 1{\cdot}042 \times 30 = 31{\cdot}26 \text{ square feet,}$$

and therefore Q becomes:

$$Q = 31{\cdot}26 \times 5{\cdot}78 = 180{\cdot}5 \text{ cu. ft. per second,}$$

which agrees with the result found directly from the general formula.

8. Belanger's formula — Sudden enlargement. — Loss of head. Up to this point we have considered that the various stream-lines which make up the whole cross-section of the flowing water are independent of one another, and that each follows its own path as if it were alone, that is to say without being affected by the presence of the neighbouring stream-lines. This amounts to the same thing as neglecting the possible friction of one stream-line on

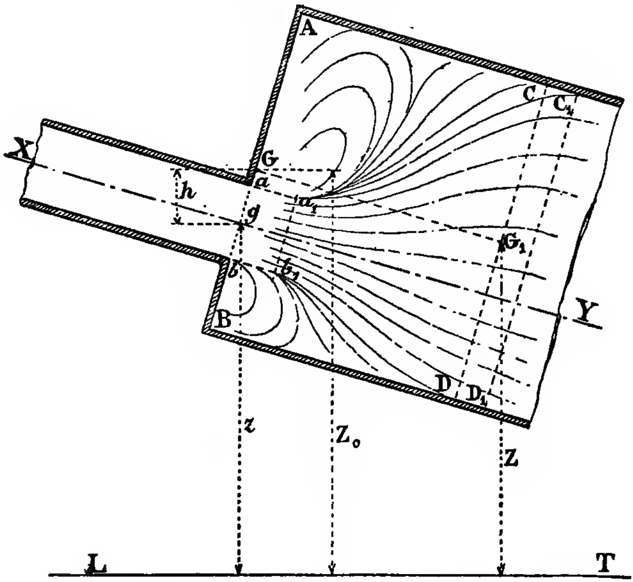

Fig. 5. Sudden enlargement of section.

another or the shocks between the molecules, both of which causes may under certain conditions very considerably modify the flow.

We will now pass on to examine the various circumstances in which friction may be of sufficient importance to make it impossible to neglect it without arriving at totally inaccurate results.

The most interesting of these cases to study is that which occurs when water flowing in a pipe suddenly enters a channel of very much larger section, without progressive transition by insensible degrees; such an arrangement is represented in fig. 5.

At the mouth of the conduit on the left the section of the stream suddenly changes from the value *ab* to the value AB.

The molecules are not guided after leaving *ab* and they spread out in all directions to fill the enlarged space, in doing so they rub against one another and, eddying about, produce shocks and vortices which tend to reduce and destroy their velocity.

As a result it happens that very close to the wall AB the water is almost at rest subject to the laws of hydrostatic pressure.

At a certain distance from this side AB, the stream-lines have regained their regular orientation and are again flowing on in parallel lines across some such section as CD; in this section also the flow obeys the laws of hydrostatics.

Hence the stream is composed of independent stream-lines in the ordinary way at the sections *ab* and CD, but in the intervening space a turmoil is produced, in consequence of which each molecule of the liquid mass has a different velocity, irregular in character and impossible to analyse or calculate.

As a consequence it is not possible to apply the considerations which served as the basis on which the Bernoulli formula was built up. When considering that, we found that the change in the kinetic energy, or the energy due to the velocity, between two sections was represented by the expression

$$\frac{q \times d}{g} \times \frac{V^2 - V_0^2}{2}.$$

This is the product of the mass $\frac{q \times d}{g}$ and half the difference of the squares of the velocities in the two sections

considered. That is to say we implicitly assumed that the stream-lines were regular and parallel, and that the velocity, or the square of the velocity, was the same for all molecules at any particular section, and that it changed in a continuous and progressive manner from the value V_0 up to the value V.

But now this is no longer the case because of the disturbance between the two sections under consideration.

We must therefore have recourse to another principle which does not assume, as formerly, that the speed changes gradually from V_0 to V, but depends only on the velocities at the particular instants when the stream crosses the sections *ab* and CD respectively.

This principle is based on the consideration of the quantity of movement, or the momentum possessed by a material body. This quantity is measured by the product of the mass of the body and its velocity.

All forces acting on a body impress upon it a tendency to accelerate, i.e. an impulse which increases the velocity and the momentum of the body. This impulse is necessarily proportional to the magnitude of the force and to the time during which it acts; so that its value is obtained by multiplying the force by the time, or is simply the force itself if the time is equal to one second or unity.

This being so, the principle made use of may be expressed thus: **the increase of momentum of a material body in one second is equal to the vector sum of the forces acting during that time.**

The forces acting are the same as those considered formerly in Bernoulli's problem, to wit: the pressure on the section AB, the back pressure on the section CD and the weight of the mass of liquid between the two sections.

Let p_0 be the pressure per square inch on the section AB of area S_0 square inches, the total pressure on this surface will be $p_0 \times S_0$; in the same way we may put the pressure on the section CD equals $p \times S_0$.

Now the weight of the liquid mass is a vertical force which

consequently does not act along XY in the direction of movement. What we require to know is the force in this direction due to the action of gravity, and this is evidently the component of the vertical force in the direction XY.

The total weight of the mass is got by multiplying the volume by the density, or $d \times S_0 \times GG_1$. In order to find the component of this force in the direction GG_1, we may notice that if GG_1 were vertical, the total weight itself would be the proper component. That is to say the component would be obtained by multiplying dS_0 by the projection of GG_1 on itself in this particular case, or, more generally, on a vertical passing through G.

If then GG_1, instead of being vertical is inclined as in the actual figure, its projection on the vertical will be equal to $(Z_0 - Z)$, namely the difference of heights of the centres of gravity G and G_1 of the two sections AB and CD. Therefore the component sought will be:

$$d \times S_0 \times (Z_0 - Z).$$

Now let V_0 and V be respectively the velocities at the sections ab and CD, and let q be the volume flowing per second which may be represented by aba_1b_1 up stream and CDC_1D_1 down stream.

Everything is the same as if during the second considered, the intervening mass remained stationary and the prism of water aba_1b_1 was transferred to CDC_1D_1 changing in velocity from V_0 to V.

The increase of the momentum of the mass $\frac{q \times d}{g}$ during this time will be:

$$\frac{q \times d}{g} \times (V - V_0) = \frac{V \times S_0 \times d}{g} \times (V - V_0),$$

since $q = S_0 \times V$, and according to the principle enunciated above, we may write:

$$\frac{V \times S_0 \times d}{g} \times (V - V_0) = p_0 S_0 - p S_0 + d S_0 \times (Z_0 - Z).$$

Dividing throughout by the factors $S_0 \times d$, this becomes:

$$\frac{V \times (V - V_0)}{g} = \frac{p_0}{d} - \frac{p}{d} + (Z_0 - Z).$$

But $\frac{p_0}{d}$ is the pressure head or the height of the free surface level at G; if we denote that at g by $\frac{p_1}{d}$, it is evident that:

$$\frac{p_0}{d} = \frac{p_1}{d} - h,$$

and further since $Z_0 = z + h$, hence we may write in conclusion:

$$\frac{V \times (V - V_0)}{g} = \frac{p_1}{d} - \frac{p}{d} + z - Z = \frac{p_1}{d} - \frac{p}{d} + H,$$

on putting $(z - Z) = H$.

For the sake of useful comparison with Bernoulli's expression, this should be put in another form; the terms $\frac{V^2}{2g} - \frac{V_0^2}{2g}$ may be added to each member of the expression without disturbing the equality, and by transposing one side of the equation we shall get:

$$\frac{V^2 - V_0^2}{2g} = \frac{p_1}{d} - \frac{p}{d} + H - \frac{V \times (V - V_0)}{g} + \frac{V^2 - V_0^2}{2g}.$$

Now it is easy to see that:

$$\frac{V^2 - V_0^2}{2g} - \frac{V \times (V - V_0)}{g} = -\frac{(V_0 - V)^2}{2g}.$$

Supposing for instance any arbitrary values whatever such as $g = 1$, $V = 2$, and $V_0 = 1$, replacing the letters in the formula by these figures, the first member becomes:

$$\frac{4 - 1}{2} - \frac{2 \times 1}{1} = -\frac{1}{2},$$

and for the second we get:

$$-\frac{(1 - 2)^2}{2} = -\frac{1}{2},$$

proving the equality stated.

We may write therefore:

$$\frac{V^2}{2g} - \frac{V_0^2}{2g} = \frac{p_1}{d} - \frac{p}{d} + H - \frac{(V_0 - V)^2}{2g},$$

or further:

$$\frac{V^2}{2g} - \frac{V_0^2}{2g} = h_1 - h + \left(H - \frac{(V_0 - V)^2}{2g}\right).$$

This is Belanger's formula, and it is seen to differ from that of Bernoulli by the term $-\frac{(V_0 - V)^2}{2g}$ subtracted. This term is the imaginary height equivalent to the loss of head between the two sections *ab* and CD.

This imaginary height is subtracted from H, so that the result is just the same as if the centre of gravity G_1 had been raised by an amount equal to $\frac{(V_0 - V)^2}{2g}$ and there had been no sudden enlargement.

It follows therefore that the difference of heights $\frac{V^2}{2g} - \frac{V_0^2}{2g}$ which constitute the head to which the production of the current is due, is reduced by this quantity, and it is because of this that the expression $\frac{(V_0 - V)^2}{2g}$ is called the loss of head due to sudden enlargement of section.

CHAPTER II.

FLOW OF LIQUIDS IN DELIVERY PIPES.

9. Loss of energy by friction. Formulae of Prony and Darcy. Except in the particular case with which we have just been dealing in which in consequence of a sudden enlargement in the pipe, the stream-lines diverged from their general direction and were subjected to friction which momentarily disturbed the continuity of their movement and modified the velocity, we have hitherto neglected disturbances due to frictional forces.

When however these affect the transmission of liquids, and especially in the distribution of water through delivery pipes, it is necessary to take them into account.

Everybody knows that whenever either a solid or a liquid moves past another fixed body by sliding over it, an opposing force is set up, due to friction, which tends to retard the motion of the moving body; this force acts then on the moving body in the opposite sense to its direction of movement, and is exerted in the plane of contact where the sliding takes place.

This happens with water flowing past the inner surfaces of a pipe, the outer surface of the cylinder of water rubs against these sides with which it is in contact, and its velocity becomes reduced in consequence: the water within slides in turn on this outer layer, and the outer layers of this are checked in the same way but not to the same extent as the outermost layers of all which are in contact with the solid sides.

In fact one may imagine the whole of the fluid to be formed of thin concentric layers or tubes, which slide one inside another like the tubes of a telescope, the outer sheet being most retarded and the stream-line in the axis of the cylinder having the greatest velocity.

It is impossible to calculate the total retardation resulting from friction either between the outer layer and the sides, or between the different layers themselves, and it is only possible to construct formula based on experimental results which apply to certain conditions with which we will now deal.

Let us consider a portion of a conduit of length l, and apply the Bernoulli formula to the prism of water between the two sections AB and CD of this piece (fig. 6).

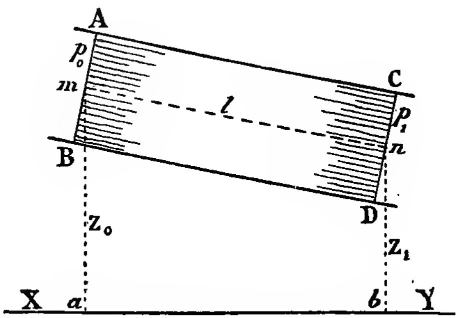

Fig. 6. Steady flow in a pipe.

In order to do this we must know the velocity of each of the stream-lines passing the sections AB and CD. These stream-lines are more and more retarded from the centre to the periphery, and have all of them different velocities in consequence. But it is easy to surmount this difficulty by considering the mean square of the velocity.

What we most require to know is the discharge or the volume of liquid flowing per second; now the discharge will be the same whether all the stream-lines have different velo-

cities or whether there is one velocity, the mean velocity which is common to them all.

Discharge Q = section S × mean velocity u.

The mean velocity is determined then by the condition that it would give the same discharge for the given section S.

From the expression :

$$Q = S \times u,$$

we get :

$$u = \frac{Q}{S}.$$

And hence we may obtain the mean velocity u_0 at AB and the mean velocity u_1 at CD.

*We can now apply Bernoulli's theorem :

$$\frac{u_0^2}{2g} + \frac{p_0}{d} + Z_0 = \frac{u_1^2}{2g} + \frac{p_1}{d} + Z_1.$$

But experiment proves that this formula does not give correct results when applied to such a case as we are considering, for it takes no account of the considerable friction which is set up in the conduit.

Bernoulli's formula indicates that, neglecting friction, to obtain a speed u_1 at the section CD, it is necessary to have at AB a speed u_0, a pressure p_0 and an altitude Z_0, so that the sum of the energy $\frac{u_0^2}{2g} + \frac{p_0}{d} + Z_0$, shall be equal to the sum of the energy corresponding to the velocity u_1 at the section CD.

But the total energy at AB given by the formula is not sufficient, when it becomes necessary to take into account the friction which is exerted throughout the length of the conduit; this height must be augmented by an additional amount which shall be sufficient to overcome, and is therefore equivalent to, the sum of the opposing frictional forces.

Let us use F to denote this resultant force per foot length

* Note by Translator. The assumption is made here that the mean square of the velocity *per unit mass* at any section is equal to the square of the mean velocity at that section, which is of course not strictly true.

of the conduit. It is easy to calculate the height of the column of liquid equivalent to this force; let k be this height; the volume of the column of water having for base the section S of the conduit will be $S \times k$, and its weight will be $S \times k \times d$; it is this weight which must be equivalent to the force F. Therefore we may write:

$$F = S \times k \times d,$$

from which we deduce:

$$k = \frac{F}{S \times d}.$$

The total force of friction for the whole length l of the pipe will be $F \times l$, and the corresponding height will be l times as great, or equal to $k \times l = K$. The height K will be the supplementary height which it is necessary to introduce into the Bernoulli formula in order to obtain the desired equilibrium between the forces producing the motion and the forces of friction which tend to retard it.

Therefore we will now modify the formula as follows:

$$\frac{u_0^2}{2g} + \frac{p_0}{d} + Z_0 = \frac{u_1^2}{2g} + \frac{p_1}{d} + Z_1 + K.$$

The supplementary height K, by which the energy must be increased at the up-stream section AB, constitutes what is called the loss of head due to friction. If then we use H_0 to denote the total energy at AB expressed as a head, and H_1 at CD, the above expression may be written:

$$H_0 = H_1 + K, \text{ or } H_1 = H_0 - K.$$

Consequently in passing from the section AB to the section CD, the head is diminished by K, becoming $(H_0 - K)$.

The value of this loss of head K may be obtained from a formula discovered by Prony, which is expressed as follows:

$$F \times l = l \times C \times (m \,.\, u + n \,.\, u^2).$$

In this formula l is the length of the conduit under consideration, C is the perimeter of the inside of the conduit, and u is the mean velocity of flow which is the same at every section if the bore be uniform.

As we have seen already:

$$K = \frac{F \times l}{S \times d}.$$

Hence: $$K = \frac{F \times l}{S \times d} = \frac{l \times C}{S \times d} \times (m \,.\, u + n \,.\, u^2).$$

But this expression may be put in a simpler form:

$$K = \frac{l \times C}{S} \times \left(\frac{m}{d} \,.\, u + \frac{n}{d} \,.\, u^2\right).$$

For, instead of dividing the sum, we may divide each of the terms by d; now putting $\frac{m}{d} = a$ and $\frac{n}{d} = b$, the expression becomes:

$$K = \frac{l \times C}{S} \times (a \,.\, u + b \,.\, u^2).$$

The factors or coefficients a and b have been determined by Prony as follows:

$$a = 0{\cdot}000018 \text{ and } b = 0{\cdot}000106,$$

where K, l and C are expressed in feet, u in feet per second, and S is in square feet.

As Darcy has pointed out, the coefficients given by Prony do not take any account of the state of the inner surfaces of the pipes; for such pipes as are used ordinarily for the supply of water, he suggests the higher coefficients:

$$a_1 = 0{\cdot}000032 \text{ and } b_1 = 0{\cdot}000135.$$

If the velocity of the water u were very small, one could neglect the term u^2, for the square of a small fraction is itself very much smaller. But in practice, it is advisable to construct conduits of such dimensions that the speed does not fall below 6 inches per second; for this minimum velocity the two terms $a \,.\, u$ and $b \,.\, u^2$ do not differ from each other to such an extent that one is negligible as compared with the other, for approximately, using Prony's figures, the relation between the two quantities is:

$$\frac{a \,.\, u}{b \,.\, u^2} = \frac{1}{3}.$$

The term $b \, . \, u^2$ is then greater than $a \, . \, u$, but when the velocity u exceeds 3 feet per second, $a \, . \, u$ begins to be negligible in comparison with $b \, . \, u^2$, for with this value of u, we have:

$$\frac{a \, . \, u}{b \, . \, u^2} = \frac{10}{177},$$

so that $a \, . \, u$ is less than one-seventeenth of $b \, . \, u^2$; and for higher velocities we may in consequence simplify the formula for K, thus:

$$K = \frac{l \times C}{S} \, . \, b \, . \, u^2.$$

In order to allow for the term omitted, b may be taken as 0·0001172, according to Dupuit.

The loss of head per foot will be simply:

$$k = \frac{C}{S} \, . \, b \, . \, u^2,$$

and in the general case:

$$k = \frac{C}{S} \times (a \, . \, u + b \, . \, u^2).$$

10. Sudden and successive changes of section. Loss of head due to sudden enlargement in the section of conduits, is calculated according to the theory already dealt with.

The accompanying sketch illustrates various cases which may occur in practice (fig. 7).

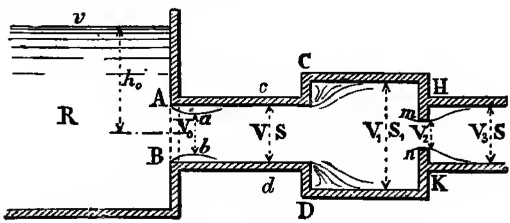

Fig. 7. Conduits of various sections.

First a conduit leaves the reservoir R, and on entering this conduit the water contracts to the section ab, and then

afterwards expands to fill the full section of the pipe; this phenomenon is equivalent to a sudden enlargement on leaving the contracted section and gives rise to a loss of head on this account.

Belanger's formula which we reproduce here:

$$\frac{V^2}{2g}-\frac{V_0^2}{2g}=h_1-h+\left(H-\frac{(V_0-V)^2}{2g}\right)$$

may be considerably simplified in this case.

V_0 is the velocity at the contracted section, and V that further along the pipe; $h_1=\frac{p_1}{d_1}$ and $h=\frac{p}{d}$ are respectively the hydrostatic pressures at the same sections, and finally H, which is the difference in heights between the centres of gravity of the two sections, is nothing in this case, since the axis of the pipe is assumed to be horizontal. Transposing the terms we may write them:

$$\frac{V^2}{2g}+\frac{(V_0-V)^2}{2g}=\frac{V_0^2}{2g}+\frac{p_1}{d}-\frac{p}{d}.$$

If now we put h_0 equals the height of the constant level in the reservoir above the axis of the pipe, Bernoulli's theorem may be applied connecting the upper level and the contracted section, and will give us:

$$\frac{v^2}{2g}+h_0+\frac{p_a}{d}=\frac{V_0^2}{2g}+\frac{p_1}{d}.$$

And since the velocity v at the surface is zero:

$$h_0=\frac{V_0^2}{2g}+\frac{p_1}{d}-\frac{p_a}{d}.$$

Finally in the case in which the back pressure exerted on section *cd* is equal to the atmospheric pressure p_a, we shall have:

$$\frac{p_a}{d}=\frac{p}{d},$$

from which: $$h_0=\frac{V_0^2}{2g}+\frac{p_1}{d}-\frac{p}{d}.$$

Consequently Belanger's formula becomes:

$$\frac{V^2}{2g}+\frac{(V_0-V)^2}{2g}=h_0.$$

If the assumption had been made at first that the same pressure was exerted on the free surface of the reservoir as on the down-stream section of the tube *cd*, this formula could have been written directly.

h_0 really constitutes the total effective head since the two equal and contrary pressures, whether equal to or greater than atmospheric pressure, equilibrate each other and therefore cancel. This height h_0 must therefore be equal to the sum of the heights $\frac{V^2}{2g}$ and $\frac{(V_0-V)^2}{2g}$ corresponding to the velocity head on the one hand, and to the loss of head due to sudden enlargement on the other.

We know that the contracted section equals 0·62 of the total section of the conduit; the speeds in the two sections are therefore in inverse ratio, that is:

$$V=0{\cdot}62\,V_0,$$

from which we may deduce:

$$V_0-V=0{\cdot}38\,V_0 \text{ and } (V_0-V)^2=0{\cdot}38^2V_0^2,$$

and inserting this value in the expression for h_0, we find:

$$h_0=\frac{V^2}{2g}+0{\cdot}38^2\frac{V_0^2}{2g},$$

and since conversely we have:

$$V_0=\frac{V}{0{\cdot}62},$$

it follows that:

$$h_0=\frac{V^2}{2g}+\frac{V^2}{2g}\left(\frac{0{\cdot}38}{0{\cdot}62}\right)^2=\frac{V^2}{2g}+0{\cdot}376\,\frac{V^2}{2g}.$$

The value h_0 in terms of the speed V the velocity at *cd* will be then:

$$h_0=1{\cdot}376\,\frac{V^2}{2g},$$

and conversely:

$$\frac{V^2}{2g} = \frac{1}{1{\cdot}376} h_0 = 0{\cdot}73 h_0.$$

Actually the speed V is less than is obtained from this expression because of the friction against the walls of the tube, of which we have taken no account, and it has been found by experiment that:

$$\frac{V^2}{2g} = 0{\cdot}67\, h_0 \text{ and } h_0 = \frac{1}{0{\cdot}67} \frac{V^2}{2g}.$$

Returning now to the Belanger formula in the simplified form which holds for the particular case under consideration, we can write the value of the loss of head:

$$\frac{(V_0 - V)^2}{2g} = h_0 - \frac{V^2}{2g},$$

and as a result of the relation established just above:

$$\frac{(V_0 - V)^2}{2g} = h_0 - 0{\cdot}67 h_0 = 0{\cdot}33 h_0.$$

Thus the loss of head produced at the outlet of the reservoir, in the pipe leading away from it, is equal to one-third of the effective head h_0 on the centre of the orifice.

Again we may write:

$$\frac{(V_0 - V)^2}{2g} = \frac{0{\cdot}33}{0{\cdot}67} \cdot \frac{V^2}{2g} = \frac{1}{2} \cdot \frac{V^2}{2g}.$$

That is the loss of head is equal to half the head corresponding to the velocity V, and as a consequence, in order to obtain this velocity in the pipe, it is necessary to have an effective head h acting on the centre of the orifice equal to that necessary for transformation to $\frac{V^2}{2g}$ plus an amount $\frac{1}{2}\frac{V^2}{2g}$ or $h = 1{\cdot}5 \frac{V^2}{2g}$.

In order to calculate the losses of head produced in the other cases, we have only to make use of facts already learned.

At CD there is a sudden enlargement of section, and the

water rushes into the part CK of the conduit with eddies which absorb a certain amount of energy, and give rise to a loss of head which we have already calculated.

V and V_1 being the velocities before and after CD, this loss of head is expressed by:

$$\frac{(V-V_1)^2}{2g}.$$

Since at any instant, the volume of water passing every section is the same, we have:

$$V_1 S_1 = VS \text{ or } V_1 = V\frac{S}{S_1},$$

and therefore we may write:

$$V - V_1 = V - V \cdot \frac{S}{S_1} = V\left(1 - \frac{S}{S_1}\right).$$

From which:
$$\frac{(V-V_1)^2}{2g} = \frac{V^2}{2g}\left(1 - \frac{S}{S_1}\right)^2.$$

Finally at section HK, the water flows through a sharp-edged orifice, and we have at the same time, contraction in passing the orifice and sudden enlargement beyond.

Let V_2 stand for the velocity in the contracted section, and V_3 the velocity further on where the flow is again steady, and the stream-lines are parallel. Let the area of the latter section be denoted by S_3.

The loss of head is always dependent on the difference of velocities before and after, and is given by:

$$\frac{(V_2-V_3)^2}{2g}.$$

The continuous flow of water throughout the whole length of the pipe enables us to write:

$$0{\cdot}62 S_2 \times V_2 = S_3 V_3,$$

where S_2 is the area of the sharp-edged orifice, and therefore $0{\cdot}62 S_2$ is the area of the most contracted section.

Hence:
$$V_2 = \frac{S_3 V_3}{0{\cdot}62 S_2}.$$

Thus it is possible to obtain an expression for the loss of head in terms of the speed V_3 at the delivery end:

$$\frac{(V_2 - V_3)^2}{2g} = \frac{V_3^2}{2g} \cdot \left(\frac{S_3}{0 \cdot 62\, S_2} - 1\right)^2.$$

These then are the different losses of head which occur as the result of sudden changes of section in distribution mains.

11. Change of direction.—Various formulae. Up to the present we have supposed the conduits to be continued in a straight line, but in practice numerous changes of direction occur.

Pipe lines in different directions ought to be joined by easy bends; this is preferable to allowing the different directions to branch off from each other abruptly, in which latter case the junction is said to be made by a sharp elbow.

It is evident that at the various changes in direction, the stream-lines collide with the sides of the conduit, and as a result shocks and eddies are produced causing losses of head which will be more important where the transition is least gradual, as where the junction is made by a sharp elbow, or a bend of very small radius.

It is impossible to determine these losses of head from theoretical considerations, but empirical formulae have been established, which enable an approximate estimation to be made for either sudden or gradual change of direction.

The following formula, established by Navier, is useful for calculating loss of head in bends:

$$P = \frac{u^2}{2g}(0 \cdot 0127 + 0 \cdot 0186\, r)\frac{a}{r^2}.$$

In this formula, u is the mean velocity in feet per second over the cross-section, obtained by dividing the discharge in cubic feet per second, by the area of cross-section; r is the mean radius of curvature of the bend in feet, and a is the length of the arc of the bend in feet.

As an example, suppose that two straight lengths with an interior angle of 80 degrees are joined with a bend whose

mean radius of curvature is 3 feet, the angle embraced by the arc of the bend $= 180 - 80 = 100$. Therefore the value of a will be:

$$a = 2\pi r \times \frac{100}{360} = \frac{\pi \times r \times 100}{180} = 5{\cdot}24 \text{ ft.}$$

If we suppose a velocity of 3 feet per second, the loss of head will be:

$$P = \frac{9}{64} \cdot (0{\cdot}0127 + 0{\cdot}0186 \times 3) \frac{5{\cdot}24}{9} = 0{\cdot}0056 \text{ ft.},$$

which is equivalent to about $\frac{1}{5000}$ of the atmospheric pressure, and is a quite negligible quantity.

Even with a relatively sharp interior angle of 40°, for example, and a radius of curvature of 1 foot, the fall of pressure for the same velocity will only be:

$$P = \tfrac{9}{64}\,(0{\cdot}0127 + 0{\cdot}0186)\,2{\cdot}44 = 0{\cdot}01073 \text{ ft.}$$

From which we see that for bends, the loss of head is generally of very little importance; thus the total loss of head in 100 such bends would only be about $\frac{1}{30}$ of the atmospheric pressure.

There are other formulae, especially those due to Rankine, which give more exact results, for they take account of the diameter of the pipe, which must of course have an influence on the loss of head, but the formulae are much more complicated and very difficult to apply.

For sharp elbows, the loss of head depends upon a coefficient K, which is expressed thus:

$$K = 0{\cdot}946 \left(\frac{a}{c}\right)^2 + 2{\cdot}047 \left(\frac{a}{c}\right)^4.$$

In order to measure $\frac{a}{c}$, on the bisectrix of the interior angle between the two directions, any distance c is measured from the intersection, and from the extremity a perpendicular is let fall upon one side cutting off a length a.

If $\frac{a}{c} = 0{\cdot}7$, which corresponds to a right angle between the two directions, it follows that:

$$K = 0{\cdot}946 \times 0{\cdot}49 + 2{\cdot}047 \times 0{\cdot}24 = 0{\cdot}955.$$

And Rankine's formula gives as the loss of head:

$$P = K \cdot \frac{u^2}{2g} = 0{\cdot}955\ \frac{u^2}{2g},$$

so that for a velocity of three feet per second:

$$P = 0{\cdot}955 \times \tfrac{9}{64} = 0{\cdot}134\ \text{ft.};$$

and for a hundred such elbows a loss of pressure of nearly half an atmosphere would result.

We have now considered all the principal losses of head which may occur in a length of pipe.

Let us tabulate them as follows:

(1) Loss of head due to friction of liquid in straight pipes:

$$K = \frac{C \times l}{S} \times (au + bu^2).$$

(2) Loss of head due to sudden enlargement of section:

$$K_1 = \frac{(V_0 - V)^2}{2g}.$$

(3) Loss of head on leaving a reservoir and entering a conduit:

$$K_2 = \frac{1}{2} \cdot \frac{V^2}{2g}.$$

(4) Loss of head in passing suddenly from a section S to a section S_1, and from a velocity V to a velocity V_1:

$$K_3 = \frac{V^2}{2g}\left(1 - \frac{S}{S_1}\right)^2.$$

(5) Loss of head in passing suddenly from an orifice with sharp edges of section S_1 to a section S and a velocity V:

$$K_4 = \frac{V^2}{2g}\left(\frac{S}{0{\cdot}62\,S_1} - 1\right)^2.$$

(6) Loss of head in a bend:

$$K_5 = \frac{V^2}{2g} \cdot (0{\cdot}0127 + 0{\cdot}0186r)\frac{a}{r^2}.$$

(7) Loss of head in a sharp elbow:

$$K_6 = \frac{V^2}{2g}\left[0{\cdot}946 + 2{\cdot}047\left(\frac{a}{c}\right)^2\right]\left(\frac{a}{c}\right)^2.$$

In order to take account of all these losses of head, the Bernoulli formula should take the form :

$$\frac{u_0^2}{2g}+\frac{p_0}{d}+Z_0=\frac{u_1^2}{2g}+\frac{p_1}{d}+Z_1+K+K_1+K_2+K_3+K_4+K_5+K_6,$$

the quantities K_1, K_2, etc., representing the losses of head due to the different causes.

12. Calculation of the discharge through a cylindrical pipe. The formulae now established enable us to solve various problems concerning the flow of liquids in pipes.

As an example, let us examine in the first place the conditions of flow of water in *a cylindrical pipe fed from a reservoir in which the level is maintained at a constant height.*

For this purpose, we will take our numerical example from the work of M. Vigreux, *l'Art de l'ingénieur*, from which we have already quoted on several occasions.

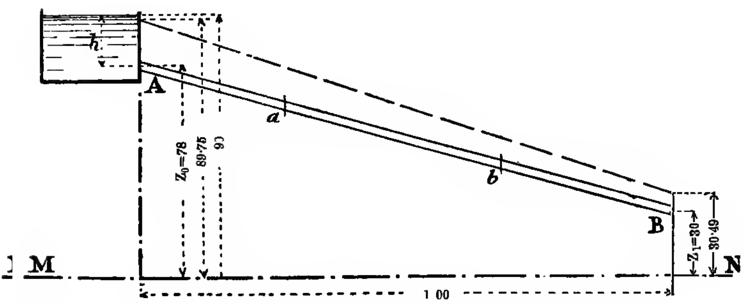

Fig. 8. Flow in a pipe.

In fig. 8 let A be a reservoir delivering water through a pipe of constant bore, let the inside diameter of the pipe be 5 inches and the length 600 yards; the level of the water in the reservoir is maintained at say 90 feet above the plane MN, suppose also that the ends of the axis of the pipe are 78 feet and 30 feet respectively above the same plane.

It is required to calculate the volume of water delivered per second by this pipe.

As in this case there are no bends or sudden changes of section, the general formula just established reduces to:

$$\frac{u_0^2}{2g} + \frac{p_0}{d} + Z_0 = \frac{u_1^2}{2g} + \frac{p_1}{d} + Z_1 + K.$$

The pipe being uniform by hypothesis, and the quantity passing every section being necessarily the same, it is evident that the mean velocity is the same throughout the length of the pipe, and not only so but also the mean square of the velocity will be the same at every point, since the same conditions hold throughout, and therefore the velocity distribution is the same everywhere.

The two terms $\frac{u_0^2}{2g}$ and $\frac{u_1^2}{2g}$ are consequently equal and cancel each other, reducing the expression to:

$$\frac{p_0}{d} + Z_0 = \frac{p_1}{d} + Z_1 + K.$$

In this formula, K denotes the loss of head due to friction in the whole length of the straight pipe, its value is:

$$K = \frac{C \times l}{S} . (a . u + b . u^2).$$

C represents the perimeter of the inside of the pipe, and S represents the sectional area; denoting the diameter in feet:

$$C = \pi D \text{ and } S = \frac{\pi D^2}{4}.$$

The term $\frac{C}{S}$, which figures in the formula, becomes in consequence:

$$\frac{C}{S} = \frac{\pi D}{\frac{\pi D^2}{4}} = \frac{4}{D}.$$

Therefore: $$K = \frac{4l}{D}(a . u + b . u^2).$$

Hence we may write:

$$\frac{p_0}{d} + Z_0 = \frac{p_1}{d} + Z_1 + \frac{4l}{D} . (a . u + b . u^2).$$

This formula can now be applied to the two end sections of the pipe A and B. It is easy to get at the value of the pressures p_0 and p_1 in these two sections.

At the lower end the pressure is obviously that of the atmosphere, so that $p_1 = p_a$.

The pressure at the upper end A just inside the reservoir, and for a height of water h above the orifice, the velocity u having been acquired would be:

$$p_0 = p_a + d \times h - \frac{u^2}{2g} \times d.$$

But as in reality we must consider the pressure in the section A, after the contraction which is produced on leaving the reservoir, it is necessary to take account of the resulting loss of head, which is equal to half the velocity head corresponding to u (§ 10) and hence:

$$p_0 = p_a + dh - \frac{1{\cdot}5u^2}{2g} \times d.$$

The formula finally becomes:

$$\frac{p_a}{d} + h - \frac{1{\cdot}5u^2}{2g} + Z_0 = \frac{p_a}{d} + Z_1 + \frac{4l}{D}(a . u + b . u^2),$$

from which, cancelling the common terms and transposing:

$$\frac{4l}{D}(a . u + b . u^2) + \frac{1{\cdot}5u^2}{2g} = h + Z_0 - Z_1.$$

The second member represents the difference of level between the surface of the water in the reservoir and the lower end of the pipe, or in other words, the total effective head. This head is used up in overcoming friction, and in generating the velocity, the two parts into which it is divided being expressed respectively by the two terms of the first member of the equation.

This formula is rather complex and would require for its solution, that is for the deduction of u, the use of algebra.

But it can be solved indirectly, by ordinary arithmetical methods, if recourse is had to the second formula giving the value of K (§ 9), namely:

$$K = \frac{l \times C}{S} \cdot b \cdot u^2.$$

Putting this simplified value in the preceding equation, it takes the form:

$$\frac{4l}{D} \cdot b \cdot u^2 + \frac{1 \cdot 5 u^2}{2g} = h + Z_0 - Z_1,$$

and further:

$$\left(\frac{4l}{D} \times b + \frac{1 \cdot 5}{2g}\right) u^2 = h + Z_0 - Z_1,$$

whence:

$$u = \sqrt{\frac{h + Z_0 - Z_1}{\frac{4lb}{D} + \frac{1 \cdot 5}{2g}}}.$$

According to a formula by Darcy, the coefficient b may be expressed thus:

$$b = 0 \cdot 000155 + \frac{0 \cdot 00000647}{r}.$$

And in the example:

$$r = \frac{D}{2} = 0 \cdot 208 \text{ ft.}$$

From which:

$$b = 0 \cdot 000155 + \frac{0 \cdot 00000647}{0 \cdot 208} = 0 \cdot 000186.$$

So that replacing the letters in the formula by their values:

$$u = \sqrt{\frac{60}{\frac{4 \times 1800}{0 \cdot 416} \times \cdot 000186 + \frac{1 \cdot 5}{64}}} = 4 \cdot 3 \text{ feet per second.}$$

We easily obtain therefore the desired result, but this simplified formula for the loss of head K, applies only to the mean speeds corresponding to pipe diameters between 8 and 16 inches.

We ought therefore to return to the complete formula in the other cases for a more exact result. It is now possible to make use of the approximate solution obtained by the

simplified formula, and to use this in what is called the method of successive approximations.

Returning then to the formula:

$$\frac{4l}{\mathrm{D}}(a \,.\, u + b \,.\, u^2) + \frac{1{\cdot}5u^2}{2g} = h + \mathrm{Z}_0 - \mathrm{Z}_1.$$

As the first value of u obtained is higher than 3 feet per second the term in u^2 preponderates very much in comparison with that in u, therefore we can put for u the value found, so that the only remaining unknown is u^2.

Hence:

$$\frac{4l}{\mathrm{D}}(a \times 4{\cdot}3) + \left(\frac{4l}{\mathrm{D}} \,.\, b + \frac{1{\cdot}5}{2g}\right) u^2 = h + \mathrm{Z}_0 - \mathrm{Z}_1.$$

And replacing the letters by their values in the particular case under consideration, we have (§ 9):

$$\frac{4 \times 1800}{0{\cdot}416}(0{\cdot}000018 \times 4{\cdot}3) + \left(\frac{4 \times 1800}{0{\cdot}416} \times 0{\cdot}000106 + \frac{1{\cdot}5}{64}\right) u^2 = 60.$$

And on making the calculations:

$$1{\cdot}34 + 1{\cdot}85u^2 = 60,$$

or: $$1{\cdot}85u^2 = 58{\cdot}67,$$

from which: $$u^2 = 31{\cdot}7,$$

and finally: $$u = 5{\cdot}6 \text{ feet per second.}$$

One might recommence the calculation introducing this new value of u into the formula, but it would be found that the result would not differ sensibly from the above, and in general a single calculation suffices for a very fair approximation.

Actually in this instance the exact value is the same to two significant figures, namely 5·6 feet per second.

The volume Q discharged per second, which was required, may now be calculated at once:

$$\mathrm{Q} = \mathrm{S} \times u = \frac{\pi \mathrm{D}^2}{4} \times 5{\cdot}6 = 0{\cdot}76 \text{ cubic feet per second,}$$

or say about 4·75 gallons per second.

The dotted line in the figure whose extremities are respectively at heights of 90 feet and 30·49 feet above the plane of

comparison MN, passes through heights which everywhere represent the total energy of the water in the pipe.

If at any such points as a or b, perpendiculars be drawn to MN and continued upwards till they meet the dotted line, the total height of the perpendicular may be denoted by H, and will be made up of two parts; the part between a or b and MN is simply the altitude of the point which we will call Z and which represents the energy in the water due to its altitude, and the part of the perpendicular beyond a or b which is included between the point considered and the dotted line, and which we will call t, represents the velocity head plus the pressure head, thus:

$$H = t + Z,$$

and
$$t = \frac{u^2}{2g} + \left\{\frac{4l_1}{D}(a \, . \, u + b \, . \, u^2) - (Z - Z_1)\right\},$$

so that:
$$H = \frac{u^2}{2g} + \frac{4l_1}{D}(a \, . \, u + b \, . \, u^2) + Z_1.$$

In this formula, $\frac{u^2}{2g}$ is the velocity head or the height corresponding to the energy in the water in virtue of its velocity u; and the second term is the head absorbed by friction in the length l_1 of the pipe between the point considered and the lower extremity B.

With regard to this last point, l_1 being zero, H becomes simply:

$$H = \frac{u^2}{2g} + Z = \cdot 49 + 30 = 30 \cdot 49 \text{ feet},$$

a height equal to that shown in the figure.

At the upper end, at the outlet of the reservoir, we have obviously:

$$H = Z_0 + h = 78 + 12 = 90 \text{ feet}.$$

But after the contracted section, it is necessary to subtract the loss of head due to this contraction, the value of which is:

$$\frac{0 \cdot 5u^2}{2g} = 0 \cdot 25 \text{ feet},$$

whence:
$$H = 90 - 0 \cdot 25 = 89 \cdot 75 \text{ feet},$$

which gives the height of the dotted line at the point where the pipe leaves the reservoir.

This line is therefore straight with its ends 89·75 and 30·49 feet above the plane of comparison MN.

For, if in the equation for t:

$$t=\frac{u^2}{2g}+\left\{\frac{4l_1}{D}(a\,.\,u+b\,.\,u^2)-(Z-Z_1)\right\},$$

we subtract $\frac{u^2}{2g}$ from each side, we get:

$$t-\frac{u^2}{2g}=\left.\frac{4l_1}{D}(a\,.\,u+b\,.\,u^2)-(Z-Z_1)\right\}.$$

The first member represents the pressure head, or the total height between the axis of the pipe and the dotted line less the constant height:

$$\frac{u^2}{2g}=0{\cdot}49.$$

These heights thus reduced are contained between the dotted line and an imaginary line drawn parallel to the axis of the pipe through the point 30·49 feet in height on the dotted line.

If we let v represent these heights thus reduced, so that:

$$v=t-\frac{u^2}{2g},$$

and if at the same time we put:

$$\frac{4}{D}(a\,.\,u+b\,.\,u^2)-\frac{(Z-Z_1)}{l_1}=K,$$

where K is a constant since it is the product of invariable quantities, we may write:

$$v=K\times l_1.$$

Now this expression shows that the amount by which the dotted line lies above the imaginary line drawn parallel to the axis of the pipe, is proportional to l_1 the distance along this line measured from the lower end; from this it follows that the slope or inclination of the dotted line with regard to the imaginary line is uniform, a fact which characterises a straight line.

Therefore it is quite sufficient to make calculations for the two ends of the pipe, as we have done, in order to get, with one operation, the straight line which, joining the two points, passes through the free surface levels of every point of the pipe.

This line is interesting as showing how the pressure falls off progressively from the point at which the pipe leaves the reservoir down to the lower end of the pipe.

13. Calculation for a conical pipe equivalent to the cylindrical one. The dotted line which forms the energy diagram in the last example has a considerable slope, and falls rapidly throughout the length of the cylindrical couduit; this would no longer be the same and the curve of the diagram would fall more gradually, over about four-fifths of its length, if one replaced the cylindrical pipe of constant diameter by a conical pipe whose diameter at the source was greater than 5 inches, and at the lower end was lower than this figure.

In the last 100 yards or so of its length, where the diameter would be relatively small, the loss of head due to friction would increase and the curve would fall rapidly towards B.

. The uniformly conical pipe has more theoretical than practical interest. From the latter point of view it might be replaced by a series of parallel lengths of decreasing diameters, having such standard dimensions as are in common use, placed end to end and joined with ordinary connections.

If we knew the dimensions of the conical pipe fulfilling the same conditions as regards quantity of water delivered, as the cylindrical one, we should obtain the different diameters of the lengths of parallel piping by adopting for the two end lengths the standard dimensions nearest the end dimensions of the conical pipe, and for the intermediate pieces, the series of standard sizes comprised between the two first chosen.

In order to avoid complicated formulae and laborious cal-

culations, we may first fix the relation between the diameters of the end sections of the conical pipe; say for example:

$$D_1 = 2D_2.$$

Then we will fix on diameters such that the mean is equal to the diameter of a cylindrical pipe of constant section D, say 6 inches, then we have:

$$\frac{D_1 + D_2}{2} = 6 \text{ or } \frac{3D_2}{2} = 6 \text{ inches or } 0{\cdot}5 \text{ ft.}$$

It follows that:

$$D_2 = 4 \text{ inches, and } D_1 = 8 \text{ inches.}$$

This conical pipe being supposed approximately equivalent to a cylindrical pipe, we will replace it by the one built up of lengths of piping of different diameters; we will fix the number of lengths at 5 say, the total length being 600 yards.

The diameters of these lengths will be chosen according to standard sizes, say:

$$d_5 = 4 \text{ inches, } d_4 = 5,\ d_3 = 6,\ d_2 = 7, \text{ and } d_1 = 8 \text{ inches.}$$

To obtain the lengths of the separate pieces which when joined together replace the uniformly conical tube, we will make use of the general formula:

$$\frac{u_1^2}{2g} + \frac{p_0}{d} + Z_0 = \frac{u_5^2}{2g} + \frac{p_1}{d} + Z_1 + K,$$

where u_1, p_0 and Z_0 represent respectively the velocity, pressure and altitude just inside the first length of piping, and u_5, p_1 and Z_1 represent the similar quantities at the delivery end.

Replacing p_0 and p_1 by their values as before:

$$p_0 = p_a + d \times h - \frac{1{\cdot}5u_1^2}{2g} \times d,$$

and:

$$p_1 = p_a,$$

the formula becomes:

$$\frac{u_1^2}{2g} - \frac{u_5^2}{2g} - \frac{1{\cdot}5u_1^2}{2g} + (h + Z_0 - Z_1) = K.$$

Now let us put the difference of level between the surface in the reservoir and the lower end of the pipe, namely,

$$(h + Z_0 - Z_1) = H,$$

for simplicity, then:

$$H - \frac{0{\cdot}5u_1^2}{2g} - \frac{u_5^2}{2g} = K,$$

or:

$$H - \frac{0{\cdot}5u_1^2}{2g} - \frac{u_5^2}{2g} - K = 0.$$

In this formula K represents the sum of all the losses due to sudden diminutions of section and also to friction throughout.

The general expression for the loss of head due to sudden diminution of section is:

$$k' = \frac{0{\cdot}5u^2}{2g} = \frac{u^2}{4g},$$

and that for loss of head due to friction is:

$$k'' = \frac{4l}{d}(au + bu^2).$$

Hence *the sum of the losses due to sudden diminution* is:

$$K' = \frac{u_2^2 + u_3^2 + u_4^2 + u_5^2}{4g},$$

and *the sum of the friction losses* is:

$$K'' = 4\left\{\frac{l_1}{d_1}(au_1 + bu_1^2) + \dots + \frac{l_5}{d_5}(au_5 + bu_5^2)\right\},$$

in which formulae the value of the suffix indicates to which of the five lengths of piping the terms refer. And $K = K' + K''$.

Now any of the velocities u may be expressed in terms of the delivery Q and the corresponding diameter in that section, for in all cases:

$$u = \frac{Q}{S},$$

and since the area of section $S = \frac{\pi d^2}{4}$, it follows that:

$$u = \frac{4Q}{\pi d^2},$$

also the square of the velocity is given by:

$$u^2 = \frac{16Q^2}{\pi^2 d^4}.$$

Therefore replacing u and u^2 by these terms:

$$K' = \frac{4Q^2}{\pi^2 g}\left(\frac{1}{d_2^4} + \frac{1}{d_3^4} + \frac{1}{d_4^4} + \frac{1}{d_5^4}\right)$$

and:

$$K'' = \frac{16aQ}{\pi}\left(\frac{l_1}{d_1^3} + \frac{l_2}{d_2^3} + \frac{l_3}{d_3^3} + \frac{l_4}{d_4^3} + \frac{l_5}{d_5^3}\right) + \frac{64bQ^2}{\pi^2}\left(\frac{l_1}{d_1^5} + \dots + \frac{l_5}{d_5^5}\right).$$

Returning now to the general formula:

$$H - \frac{0{\cdot}5u_1^2}{2g} - \frac{u_5^2}{2g} - K = 0,$$

and carrying out the same replacing operation:

$$H - \frac{4Q^2}{\pi^2 g}\cdot\frac{1}{d_1^4} - \frac{8Q^2}{\pi^2 g}\cdot\frac{1}{d_5^4} - K = 0.$$

It will be noticed that the second and third terms of this expression are similar to the terms composing K′, the third in particular being just double the last term in the expression for K′, so that replacing K in the last formula, we have:

$$H - \frac{4Q^2}{\pi^2 g}\left(\frac{1}{d_1^4} + \frac{1}{d_2^4} + \frac{1}{d_3^4} + \frac{1}{d_4^4} + \frac{3}{d_5^4}\right) - \frac{16aQ}{\pi}\left(\frac{l_1}{d_1^3} + \frac{l_2}{d_2^3} + \frac{l_3}{d_3^3} + \frac{l_4}{d_4^3} + \frac{l_5}{d_5^3}\right)$$
$$- 64\frac{bQ^2}{\pi^2}\left(\frac{l_1}{d_1^5} + \frac{l_2}{d_2^5} + \frac{l_3}{d_3^5} + \frac{l_4}{d_4^5} + \frac{l_5}{d_5^5}\right) = 0.$$

In this formula everything is known except the lengths l, the quantity Q being taken as 1·21 cubic feet per second, the quantity which would flow through a pipe 6 inches in diameter under conditions similar in other respects to those holding in the last example (§ 12).

But the total length L = 1800 feet = the sum of the lengths l_1, l_2, l_3, l_4 and l_5, and we may further fix the condition that the last four lengths shall be equal:

$$l_2 = l_3 = l_4 = l_5,$$

and therefore:

$$l_1 = L - 4l_2.$$

The general expression then becomes:

$$\mathrm{H}-\frac{4\mathrm{Q}^2}{\pi^2 g}\left(\frac{1}{d_1^4}+\frac{1}{d_2^4}+\frac{1}{d_3^4}+\frac{1}{d_4^4}+\frac{3}{d_5^4}\right)-\frac{16a\mathrm{Q}}{\pi}\left(\frac{1}{d_2^3}+\ldots+\frac{1}{d_5^3}\right)l_2$$
$$-\frac{16a\mathrm{Q}}{\pi}\cdot\frac{\mathrm{L}-4l_2}{d_1^3}-\frac{64b\mathrm{Q}^2}{\pi^2}\left(\frac{1}{d_2^5}+\ldots+\frac{1}{d_5^5}\right)-\frac{64b\mathrm{Q}^2}{\pi^2}\cdot\frac{\mathrm{L}-4l_2}{d_1^5}=0,$$

and on grouping the terms in L and l_2:

$$\mathrm{H}-\frac{4\mathrm{Q}^2}{\pi^2 g}\left(\frac{1}{d_1^4}+\frac{1}{d_2^4}+\frac{1}{d_3^4}+\frac{1}{d_4^4}+\frac{3}{d_5^4}\right)-\mathrm{L}\left(\frac{16a\mathrm{Q}}{\pi d_1^3}+\frac{64b\mathrm{Q}^2}{\pi^2 d_1^5}\right)$$
$$=\frac{16a\mathrm{Q}}{\pi}\left(\frac{1}{d_2^3}+\ldots+\frac{1}{d_5^3}-\frac{4}{d_1^3}\right)l_2+\frac{64b\mathrm{Q}^2}{\pi^2}\left(\frac{1}{d_2^5}+\ldots+\frac{1}{d_5^5}-\frac{4}{d_1^5}\right)l_2.$$

Therefore l_2 is the quotient of the first member of the equation divided by the factors by which it is multiplied in the second member:

$$l_2=\frac{\mathrm{H}-\frac{4\mathrm{Q}^2}{\pi^2 g}\left(\frac{1}{d_1^4}+\ldots+\frac{3}{d_5^4}\right)-\mathrm{L}\left(\frac{16a\mathrm{Q}}{\pi d_1^3}+\frac{64b\mathrm{Q}^2}{\pi^2 d_1^5}\right)}{16\,\frac{a\mathrm{Q}}{\pi}\left(\frac{1}{d_2^3}+\ldots+\frac{1}{d_5^3}-\frac{4}{d_1^3}\right)+\frac{64b\mathrm{Q}^2}{\pi^2}\left(\frac{1}{d_2^5}+\ldots+\frac{1}{d_5^5}-\frac{4}{d_1^5}\right)}.$$

Now putting in place of the letters their actual values, the terms in brackets become:

$$\left\{\left(\tfrac{12}{8}\right)^4+\left(\tfrac{12}{7}\right)^4+\left(\tfrac{12}{6}\right)^4+\left(\tfrac{12}{5}\right)^4+3\left(\tfrac{12}{4}\right)^4\right\}=306,$$
$$\left\{\left(\tfrac{12}{7}\right)^3+\left(\tfrac{12}{6}\right)^3+\left(\tfrac{12}{5}\right)^3+\left(\tfrac{12}{4}\right)^3-4\left(\tfrac{12}{8}\right)^3\right\}=\ 40{\cdot}35,$$
$$\left\{\left(\tfrac{12}{7}\right)^5+\left(\tfrac{12}{6}\right)^5+\left(\tfrac{12}{5}\right)^5+\left(\tfrac{12}{4}\right)^5-4\left(\tfrac{12}{8}\right)^5\right\}=339{\cdot}2.$$

Also since $\mathrm{Q}=1{\cdot}21$ cubic feet,

$$\frac{16a\mathrm{Q}}{\pi}=\frac{16\times 0{\cdot}000018\times 1{\cdot}21}{3{\cdot}14}=0{\cdot}000111,$$

and:
$$\frac{64b\mathrm{Q}^2}{\pi^2}=\frac{64\times 0{\cdot}000106\times 1{\cdot}46}{\pi^2}=0{\cdot}001.$$

On completing the calculation it will be found that $l_2=116$ feet.

Hence it follows that $l_1=\mathrm{L}-4l_2=1800-464=1336$ feet, so that the first length is very much greater than the other four put together.

14. Distributing network. One of the most common problems is the distribution of water in a network or system of pipes, consisting of a main pipe, with a number of secondary supply pipes branching from it.

As a simple example of such a distributing system, let us consider the arrangement shown in plan in figure 9.

It comprises a reservoir A in which the level of the water is maintained at a constant height of 60 feet above a certain datum level, the depth of the water being 9 feet.

A main conduit divided into five lengths of different diameters whose extremities are at various heights above the datum level, these heights being indicated in the figure by numbers placed against them, the lengths also being similarly indicated.

And finally, supply pipes branching off at the ends of the different sections, the altitudes of their extremities, *c*, *d*, *e* and *f*, as well as their lengths being also indicated in the figure.

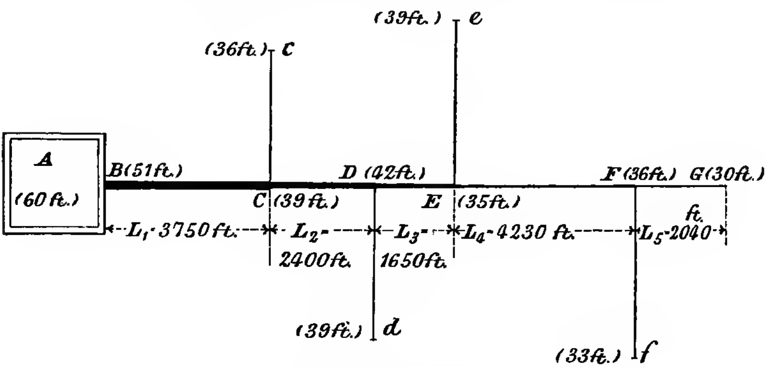

Fig. 9. Distributing system.

The problem is to determine the diameters of the successive sections of piping, in order to obtain at the end of each service pipe a definite delivery, which in this case we will take to be the same at every point, and equal to 3·12 gallons or 0·5 cubic ft.

The problem appears to be very complicated if one considers the whole system, but it is much simplified if each service is considered separately as if it were alone.

Of the five service pipes let us first consider that E*e* which is rather distant from the reservoir and at the same time has its outlet at the maximum height of 39 feet above the datum level.

If we ignore the supply pipes C*c* and D*d* which are interposed between E*e* and the reservoir it is evident that it is just as if there were only one pipe BCDE*e*, going from the reservoir to the end *e* of the service pipe considered, and having a varying flow in the different lengths from B to E.

In fact the flow in CD is reduced by the amount taken by C*c*, and in DE there is a further reduction in consequence of the amount taken by D*d*.

Except for this decreasing flow as one travels away from the upper end of the pipe, this case might be treated as an ordinary main without branching-off pipes, account being taken in the calculation of the sudden changes in section and direction.

Let us now apply the general equation to the conduit BCDE*e*, extending from the bottom of the reservoir where the extremity B is connected, to the outlet end *e*.

In the reservoir itself, on account of its large capacity, the speed may be considered quite negligible, that is the initial velocity $u_0 = 0$, and in consequence $\frac{u_0^2}{2g}$ may be at once eliminated from the formula.

Moreover if we denote the mean velocities in the different lengths L_1, L_2, L_3 and l_3 by U_1, U_2, U_3 and u_3 respectively, the final speed, which figures in the general formula as u_1 is simply u_3, the velocity in the last length.

Taking these facts into consideration, our general formula becomes:

$$\frac{p_0}{d} + Z_0 = \frac{u_3^2}{2g} + \frac{p_3}{d} + Z_1 + K.$$

In this formula $\frac{p_0}{d}$ is the head at the bottom of the reservoir where the conduit has its origin; if we use h to denote the depth of water at this point, we have:

$$\frac{p_0}{d} = \frac{p_a}{d} + h.$$

Similarly, $\frac{p_3}{d}$ is the head required at the extremity e of the pipe under consideration.

Suppose we impose the condition that the discharge at this extremity must take place under a definite head, say $h_3 = 9$ feet for example, then under these conditions, the total head will be:

$$\frac{p_3}{d} = \frac{p_a}{d} + h_3.$$

Z_0 is the height of the source B and is therefore 9 feet below the level of the water in the reservoir, or 51 feet above the datum level.

Z_1 is the height of the extremity e of the conduit, say 39 feet above the datum level.

The general formula may now be written:

$$\frac{p_a}{d} + h + Z_0 = \frac{u_3^2}{2g} + \frac{p_a}{d} + h_3 + Z_1 + K.$$

The equal terms $\frac{p_a}{d}$ cancel, moreover, the total difference of head H between the point e and the reservoir may be expressed:

$$H = h + Z_0 - (h_3 + Z_1),$$

so that replacing the corresponding terms by H, the simplified formula becomes:

$$H = \frac{u_3^2}{2g} + K.$$

That is to say, the total effective head H is employed in generating the velocity u_3 and in overcoming the various

resistances which cause loss of head throughout the whole length of the pipe.

The different losses of head which come up for consideration in this case are:

(1) Losses K_1 due to frictional resistances in the various sections of the pipe.

(2) Losses K_2 produced by the contraction at the inlet of the pipe B, and at the points C and D where the sectional area is abruptly reduced.

(3) The loss K_3 due to the sudden change of direction at E.

Expressing successively each of these losses:

$$K_1 = \frac{4L_1}{D_1}(aU_1 + bU_1^2) + \frac{4L_2}{D_2} . (aU_2 + bU_2^2)$$
$$+ \frac{4L_3}{D_3}(aU_3 + bU_3^2) + \frac{4l_3}{d_3}(au_3 + bu_3^2),$$

and (§ 10): $$K_2 = \frac{U_1^2 + U_2^2 + U_3^2}{4g},$$

and that which represents the loss of head caused by a right-angled change of direction at the point E, may be expressed according to a formula experimentally established by Belanger:

$$K_3 = \frac{3u_3^2}{2g}.$$

Putting these different values in the preceding equation:

$$H = \frac{1}{4g}(U_1^2 + U_2^2 + U_3^2 + 8u_3^2) + 4\frac{L_1}{D_1}(aU_1 + bU_1^2)$$
$$+ 4\frac{L_2}{D_2}(aU_2 + bU_2^2) + 4\frac{L_3}{D_3}(aU_3 + bU_3^2) + 4\frac{l_3}{d_3}(au_3 + bu_3^2).$$

Now we know that the unknown velocities may be expressed in terms of the known deliveries and diameters, according to the general formula:

$$u = \frac{4Q}{\pi d^2}.$$

But after this substitution, there are still four unknown quantities D_1, D_2, D_3 and d_3; now with one equation it is only possible to deduce the value of one unknown quantity, say D_1 for example, as a function of all the others, and it is evident therefore that the value of D_1 will depend on the arbitrary values assigned to these other quantities.

Moreover it may happen that the arbitrarily chosen values are incompatible with the given facts of the problem.

We will therefore consider what is the best choice to make in order to obtain a simple solution which shall satisfy all the conditions laid down.

The formula we have already obtained contains four velocities, and obviously if we could settle on a convenient relation between these four quantities, the knowledge of the value of one of them would lead to the determination of all the rest.

The most simple ratio that can be adopted is of course unity, which is the same thing as saying that the mean speed in every section is the same; let us assume then that:

$$U_1 = U_2 = U_3 = u_3.$$

This assumption, which may be introduced into the problem quite legitimately, has the advantage of making the original formula exceedingly simple.

It follows as an immediate consequence that the various diameters are proportional to the square root of the corresponding quantities carried; for between any two speeds, we have the relation:

$$\frac{4Q_1}{\pi D_1^2} = \frac{4Q_2}{\pi D_2^2},$$

and eliminating the common factors:

$$\frac{Q_1}{D_1^2} = \frac{Q_2}{D_2^2},$$

or:

$$\frac{Q_2}{Q_1} = \frac{D_2^2}{D_1^2}.$$

So that multiplying each side of the equation by D_1^2 we get:

$$D_2^2 = D_1^2 \times \frac{Q_2}{Q_1},$$

or:

$$D_2 = D_1 \times \sqrt{\frac{Q_2}{Q_1}}.$$

Similarly we have:

$$D_3 = D_1 \times \sqrt{\frac{Q_3}{Q_1}},$$

and:

$$d_3 = D_1 \times \sqrt{\frac{q_3}{Q_1}}.$$

That is to say we may express the various diameters as functions of one diameter D_1 only, which is then the only remaining unknown in the problem.

Further taking into account the condition imposed that the delivery is to be the same in each service pipe, it follows at once that if we use q to denote this common delivery:

$$q_3 = q, \quad Q_1 = 5q, \quad Q_2 = 4q, \text{ and } Q_3 = 3q.$$

Putting these values in the above expressions, they become:

$$D_2 = D_1\sqrt{\tfrac{4}{5}}, \quad D_3 = D_1\sqrt{\tfrac{3}{5}}, \text{ and } d_3 = D_1\sqrt{\tfrac{1}{5}}.$$

We can now simplify the principal equation by bringing in the assumption of the equality of the mean speeds; it then becomes:

$$H = \frac{11u^2}{4g} + 4\left(\frac{L_1}{D_1} + \frac{L_2}{D_2} + \frac{L_3}{D_3} + \frac{l_3}{d_3}\right)\{aU_1 + bU_1^2\}.$$

Expressing U_1 in terms of the common discharge q:

$$U_1 = \frac{20q}{\pi D_1^2},$$

and replacing D_2, D_3 and d_3 by their corresponding expressions in terms of D_1, we have:

$$H = \frac{1100q^2}{g\pi^2 D_1^4} + \frac{80aq}{\pi D_1^2}\left(\frac{L_1}{D_1} + \frac{L_2}{D_1\sqrt{\frac{4}{5}}} + \frac{L_3}{D_1\sqrt{\frac{3}{5}}} + \frac{l_3}{D_1\sqrt{\frac{1}{5}}}\right)$$
$$+ \frac{1600bq^2}{\pi^2 D_1^4}\left(\frac{L_1}{D_1} + \frac{L_2}{D_1\sqrt{\frac{4}{5}}} + \frac{L_3}{D_1\sqrt{\frac{3}{5}}} + \frac{l_3}{D_1\sqrt{\frac{1}{5}}}\right).$$

But arithmetical rules applied to letters just as to figures, enable us to write:

$$H = \frac{1100q^2}{g\pi^2 D_1^4} + \frac{80aq}{\pi D_1^3}\sqrt{5}\left(\frac{L_1}{\sqrt{5}} + \frac{L_2}{\sqrt{4}} + \frac{L_3}{\sqrt{3}} + l_3\right)$$
$$+ \frac{1600bq^2}{\pi^2 D_1^5}\sqrt{5}\left(\frac{L_1}{\sqrt{5}} + \frac{L_2}{\sqrt{4}} + \frac{L_3}{\sqrt{3}} + l_3\right).$$

Reducing the terms within brackets to a common denominator, they become, together with the factor $\sqrt{5}$ preceding them:

$$\sqrt{\tfrac{1}{12}}\,(L_1\sqrt{12} + L_2\sqrt{15} + L_3\sqrt{20} + l_3\sqrt{60}).$$

We may immediately calculate this value, giving the lengths the values indicated in the figure:

$$L_1 = 3750 \text{ ft.},\ L_2 = 2400 \text{ ft.},\ L_3 = 1650 \text{ ft., and } l_3 = 3720 \text{ ft.}$$

Referring to tables of square roots the above expression may be written then:

$$\frac{3750 \times 3{\cdot}464 + 2400 \times 3{\cdot}873 + 1650 \times 4{\cdot}472 + 3720 \times 7{\cdot}746}{3{\cdot}464}$$
$$= 16{,}882.$$

The principal formula may now be written as follows, putting all the terms on one side of the equation:

$$H - \frac{1100q^2}{g\pi^2 D_1^4} - \frac{80 \times 16882aq}{\pi D_1^3} - \frac{1600 \times 16882bq^2}{\pi^2 D_1^5} = 0.$$

On multiplying each term by D_1^5, which is the same thing as multiplying five times by D_1, and calculating and reducing:

$$H \times D_1^5 - \frac{1350560aqD_1^2}{\pi} - \frac{1100q^2 D_1}{g\pi^2} - \frac{27011200bq^2}{\pi^2} = 0.$$

All the quantities contained in this formula are known except the diameter D_1; in fact we are given that:

$$H = h + Z_0 - (h_3 + Z_1) = (9 + 51) - (9 + 39) = 12,$$

and:

$$a = 0{\cdot}000018;\ b = 0{\cdot}000106;\ q = 0{\cdot}5 \text{ cubic feet, } q^2 = 0{\cdot}25.$$

Replacing the letters by their values and dividing throughout by $H = 12$, we get:

$$D_1^5 - 0{\cdot}322D_1^2 - 0{\cdot}0722D_1 - 6{\cdot}03 = 0,$$

a formula containing only D_1, but this quantity raised to various powers, 5, 2 and 1.

D_1 may now be calculated by the method of successive approximations, neglecting first the terms in D_1^2 and D_1 and writing the equation simply as:

$$D_1^5 = 6{\cdot}03.$$

But if one wished to avoid the use of logarithms which would be necessary for the extraction of the fifth root, one might obtain the same result by writing the equation in the following manner, obtained by dividing throughout by D_1:

$$D_1^4 = 0{\cdot}322 D_1 + 0{\cdot}0722 + \frac{6{\cdot}03}{D_1}.$$

Now let us give any value whatever to D_1 in the second member; say $D_1 = 1$ for example; the equation then becomes:

$$D_1^4 = 0{\cdot}322 + 0{\cdot}072 + 6{\cdot}03 = 6{\cdot}4 \text{ say.}$$

Whence: $$D_1^2 = \sqrt{6{\cdot}4} = 2{\cdot}53,$$

and: $$D_1 = \sqrt{2{\cdot}53} = 1{\cdot}6.$$

Putting again this newly found value for D_1 on the second side of the equation, we have:

$$D_1^4 = 0{\cdot}322 \times 1{\cdot}6 + 0{\cdot}072 + \frac{6{\cdot}03}{1{\cdot}6} = 4{\cdot}36,$$

from which:

$$D_1^2 = \sqrt{4{\cdot}36} = 2{\cdot}08 \text{ and } D_1 = \sqrt{2{\cdot}08} = 1{\cdot}44.$$

Using this third value in the formula we find D_1 very nearly 1·47, the exact value, and it is useless to further prolong the calculation.

The value of the diameter D_1 being thus calculated, we may immediately obtain the values of the other diameters by means of the relations already set forth.

$$D_1 = 1{\cdot}47 \text{ ft.}$$

$$D_2 = 1{\cdot}47 \sqrt{\tfrac{4}{5}} = 1{\cdot}315 \text{ ft.}$$

$$D_3 = 1{\cdot}47 \sqrt{\tfrac{3}{5}} = 1{\cdot}14 \text{ ft.}$$

$$d_3 = 1{\cdot}47 \sqrt{\tfrac{1}{5}} = 0{\cdot}657 \text{ ft.}$$

These are the exact diameters fulfilling the conditions, but in practice it would be necessary to choose pipes of the standard dimensions most nearly approaching these figures.

Let us therefore replace the diameters thus:

$$D_1 = 1{\cdot}47 \text{ ft. by } 18 \text{ inches } = 1{\cdot}5 \text{ ft.}$$
$$D_2 = 1{\cdot}315 \text{ ft. by } 15 \text{ inches } = 1{\cdot}25 \text{ ft.}$$
$$D_3 = 1{\cdot}14 \text{ ft. by } 14 \text{ inches } = 1{\cdot}167 \text{ ft.}$$
$$d_3 = 0{\cdot}653 \text{ ft. by } 8 \text{ inches } = 0{\cdot}667 \text{ ft.}$$

It necessarily follows that the speed will no longer be the same in each section of the conduit, as was assumed in our calculations, in order to simplify the problem; but we may easily find the new velocities by applying for each actual diameter the general formula:

$$U = \frac{4Q}{\pi D^2}.$$

Thus the velocity in the first length is:

$$U_1 = \frac{4Q_1}{\pi D_1^2} = \frac{20q}{\pi \times 2{\cdot}25} = \frac{10}{\pi \times 2{\cdot}25} = 1{\cdot}415 \text{ feet per second.}$$

In the same way:

$$U_2 = 1{\cdot}625; \quad U_3 = 1{\cdot}40; \quad u_3 = 1{\cdot}43.$$

Let us now calculate the loss of head in each part of the system by the general formulae:

$$K_1 = \frac{4L}{D}(aU + bU^2),$$

$$K_2 = \frac{U_1^2 + U_2^2 + U_3^2}{4g},$$

and

$$K_3 = \frac{3u_3^2}{2g},$$

K_1 relating to friction against the sides, K_2 being the losses due to contraction, and K_3 the loss due to sudden change of direction.

We thus obtain successively in passing from the first section BC to the last E*e*:

$$K_1 \begin{cases} k_0 = \dfrac{4 \times 3750}{1{\cdot}5}(0{\cdot}000018 \times 1{\cdot}415 + 0{\cdot}000106 \times 1{\cdot}415^2) = 2{\cdot}38 \\ k_1 = \dfrac{4 \times 2400}{1{\cdot}25}(0{\cdot}000018 \times 1{\cdot}625 + 0{\cdot}000106 \times 1{\cdot}625^2) = 2{\cdot}38 \\ k_2 = \dfrac{4 \times 1650}{1{\cdot}167}(0{\cdot}000018 \times 1{\cdot}40 + 0{\cdot}000106 \times 1{\cdot}40^2) \quad = 1{\cdot}32 \\ k_3 = \dfrac{4 \times 3720}{0{\cdot}667}(0{\cdot}000018 \times 1{\cdot}43 + 0{\cdot}000106 \times 1{\cdot}43^2) \quad = 5{\cdot}42 \end{cases}$$

$$K_2 = \frac{1{\cdot}415^2 + 1{\cdot}625^2 + 1{\cdot}40^2}{4 \times 32} \qquad = 0{\cdot}051$$

$$K_3 = \frac{3 \times 1{\cdot}43^2}{2 \times 32} = \frac{3 \times 2{\cdot}06}{64} \qquad = 0{\cdot}096$$

Total loss of head $\quad \underline{11{\cdot}65}$

Thus the pressure diminishes on account of the loss of head, which increases from the source of the main pipe to the end of the service pipe E, and the total loss of head reaches the value 11·65 feet at the extremity *e*.

Now there was available an effective total head:

$$H = h + Z_0 - (h_3 - Z_1) = 9 + 51 - (9 + 39) = 12 \text{ ft.},$$

whereas we have only made use of 11·65 feet, therefore there is still available a head of 0·35 feet.

$$12 - 11{\cdot}65 = 0{\cdot}35.$$

Advantage may be taken of this to reduce the diameter of the pipes, in other words we have yet to expend 0·35 feet of head in friction in the pipes.

Let us make this increase of loss of head take place in the length DE by reducing its diameter over part of its length from 14 inches to 12 inches.

The speed in the reduced portion will necessarily be increased and becomes 1·91 ft. per second, so that the loss of head in friction in DE if the whole of this length were reduced in diameter to 12 inches would be:

$$k_2 = \frac{4 \times 1650}{1}(0{\cdot}000018 \times 1{\cdot}91 + 0{\cdot}000106 \times 3{\cdot}65) = 2{\cdot}38.$$

This loss was formerly 1·32 ft., so that the increase would be:

$$2{\cdot}38 - 1{\cdot}32 = 1{\cdot}06 \text{ ft.},$$

whereas it is only desired to increase the loss by 0·35 ft. We have now however sufficient data to enable us to calculate what fraction of the length DE ought to be reduced in diameter from 14 to 12 inches.

Before proceeding, however, it should be noticed that another sudden diminution of section is being introduced into the system, and the loss of head due to this is:

$$k = \frac{u^2}{4g} = \frac{3{\cdot}65}{128} = 0{\cdot}275 \text{ ft.},$$

and hence K_2, the sum of the losses due to sudden diminutions of section, is increased by this amount, and becomes therefore:

$$K_2 = 0{\cdot}0515 + 0{\cdot}0275 = 0{\cdot}079 \text{ or say } 0{\cdot}08 \text{ ft.}$$

Consequently we see that the friction in the reduced section is to be such that the resulting loss of head will be $0{\cdot}35 - 0{\cdot}0275 = 0{\cdot}32$ ft. say more than it would be in the same length, if the diameter remained 14 inches.

Now we have already seen that if the whole length of DE, namely 1650 feet, were reduced to 12 inches diameter, the increased loss of head due to friction would be 1·06 feet, and therefore since only an increased loss of 0·32 ft. is required, the length to be decreased in diameter is:

$$\frac{0{\cdot}32}{1{\cdot}06} \times 1650 = 500 \text{ ft.}$$

Therefore $1650 - 500 = 1150$ feet of the length DE is to be maintained of 14 inches diameter, and the remaining length of 500 feet will be reduced to 12 inches in diameter.

Let us definitely adopt then as corresponding to the best economic conditions of the installation, the diameters:

$D_1 = 18$ inches; $D_2 = 15$ inches; $D_3 = 14$ inches for first 1150 feet of length and 12 inches for remainder, and $d_3 = 8$ inches.

Continuing our method of splitting up the problem, let us now pass on to consider the length EFf; this is chosen in preference to the piece EFG because firstly, the extremity f is higher than G, and secondly because it is most distant from the reservoir. It follows that the effective head available for this length is less, while on account of its length, it is capable of absorbing a considerable head; it is therefore advisable to first satisfy the more exacting conditions of this piece, for it will be easy afterwards to satisfy the requisite conditions for the other piece (fig. 9).

The problem may be treated in the same manner as before; the point E plays the same part as the point B, the source in the former portion of the problem, and the point f is precisely analogous to the point e.

The available head H will be the difference in altitude, plus the head at E, less the head h_4, which must be reserved for the point f.

If there were no loss of head between the water level in the reservoir and the point E, the head at this point would be equal to the difference of their heights, that is:

$$Z_0 - Z_3 = 60 - 33 = 27 \text{ ft.}$$

But the head is reduced by all the losses due to friction and change of section, that occur between the reservoir and the point considered, these losses, as we have seen, are:

$$K_1 = 2{\cdot}38 + 2{\cdot}38 + 1{\cdot}64 = 6{\cdot}4 \text{ ft.},$$

and:
$$K_2 = 0{\cdot}08 \text{ ft.},$$

hence:
$$K_1 + K_2 = 6{\cdot}48 \text{ ft.}$$

Therefore the pressure or head at the point E will be:

$$N_3 = 27 - 6{\cdot}48 = 20{\cdot}62 \text{ ft.}$$

From E to f there is added the head due to the difference in the altitudes of these points, which in this case is nothing, since

$$Z_3 - z_4 = 33 - 33 = 0.$$

Hence the total head is simply N_3, but as the height of

the free surface level at f is $h_4 = 9$ ft., this being the head under which the outflow is required to take place, there remains as available head, only $H = 20{\cdot}62 - 9 = 11{\cdot}62$ ft.

All these facts may be gathered in one formula which may be written at once:

$$H = N_3 + (Z_3 - z_4) - h_4 = N_3 - h_4 = 11{\cdot}62 \text{ ft.}$$

In this particular case the general formula becomes:

$$H = \frac{1}{4g}(U_4^2 + 8u_4^2) + 4\frac{L_4}{D_4}(aU_4 + bU_4^2) + 4\frac{l_4}{d_4}(au_4 + bu_4^2).$$

We have also as formerly:

$$q_4 = q;\ \ Q_4 = 2q;\ \ u_4 = U_4,$$

$$d_4 = D_4\sqrt{\frac{q}{Q_4}} = D_4\sqrt{\frac{q}{2q}} = \frac{D_4}{\sqrt{2}}.$$

Whence:

$$H = \frac{9U_4^2}{4g} + 4\left(\frac{L_4}{D_4} + \frac{l_4}{d_4}\right)(aU_4 + bU_4^2).$$

And since: $$U_4 = \frac{4Q_4}{\pi D_4^2} = \frac{8q}{\pi D_4^2},$$

it follows that:

$$H = \frac{144q^2}{g\pi^2 D_4^4} + \frac{32aq}{\pi D_4^3}(L_4 + l_4\sqrt{2}) + \frac{256bq^2}{\pi^2 D_4^5}(L_4 + l_4\sqrt{2}).$$

Replacing L_4 and l_4 by their values we obtain:

$$L_4 + l\sqrt{2} = 4230 + 3960 \times 1{\cdot}414 = 9830.$$

On reducing and operating as before, the formula becomes:

$$H \times D_4^5 - \frac{32 \times 0{\cdot}000018 \times {\cdot}5}{\pi}9830D_4^2$$

$$- \frac{144 \times {\cdot}25}{32 \times \pi^2}D_4 - \frac{9830 \times 256 \times 0{\cdot}000106 \times {\cdot}25}{\pi^2} = 0.$$

Dividing by H whose value has just been determined:

$$D_4^5 - 0{\cdot}0774D_4^2 - 0{\cdot}00974D_4 - 0{\cdot}577 = 0.$$

Or writing:

$$D_4^4 = 0{\cdot}0774D_4 + 0{\cdot}00974 + \frac{0{\cdot}577}{D_4},$$

and using the method of successive approximations as before, we obtain ultimately:

$$D_4 = 0{\cdot}092 \text{ ft.},$$

whence:
$$d_4 = \frac{D}{\sqrt{2}} = 0{\cdot}65 \text{ ft.}$$

Choosing the nearest standard dimensions, let us adopt:

$D = 11$ inches or $0{\cdot}917$ ft.; $d_4 = 8$ inches or $0{\cdot}667$ ft.

The resulting speeds will be:

$$U_4 = \frac{4}{\pi\left(\frac{11}{12}\right)^2} = \frac{144 \times 4}{\pi \times 121} = 1{\cdot}515 \text{ ft. per sec.},$$

and:
$$u_4 = \frac{4 \times 0{\cdot}5}{\pi\left(\frac{2}{3}\right)^2} = \frac{4{\cdot}5}{\pi} = 1{\cdot}43 \text{ ft. per sec.}$$

Keeping to the same notation as before, the losses of head will be:

$$K_1 \begin{cases} k_0 = \dfrac{4 \times 4320}{\frac{11}{12}}(a \times 1{\cdot}515 + b \times 1{\cdot}515^2) = 5{\cdot}00 \text{ ft.} \\ k_1 = \dfrac{4 \times 3960}{\frac{2}{3}}(a \times 1{\cdot}43 + b \times 1{\cdot}43^2) = 5{\cdot}77 \text{ ft.} \end{cases}$$

$$K_2 = \frac{1{\cdot}515^2}{4g} = 0{\cdot}018 \text{ ft.}$$

$$K_3 = \frac{3 \times 1{\cdot}43^2}{2g} = 0{\cdot}095 \text{ ft.}$$

Total loss of head $\quad 10{\cdot}883$ ft.

This loss is less than the head available, and therefore as before, we may economise by reducing the section of part of either EF or F*f*. Let us choose to make the alteration in F*f*.

The calculation necessary for this is similar to that in the former part of this problem.

In order to calculate the last section of the main conduit FG, we proceed in the same manner, taking note that it comprises a single section only, and that there is no change in direction.

The pressure head at F will be:

$$N_4 = Z_0 - Z_4 - K,$$

where K denotes the sum of the losses from A to F, to wit:

$$K_1 = 6{\cdot}48 + 5{\cdot}00,$$
$$K_2 = 0{\cdot}018,$$

and hence $K = K_1 + K_2 = 11{\cdot}5$.

Also Z_0 and Z_4 are the altitudes of the reservoir and the point F respectively, therefore:

$$N_4 = 60 - 36 - 11{\cdot}45 = 12{\cdot}55.$$

The available head for the length FG will be then:

$$H = N_4 + (Z_4 - Z_5) - h_5,$$

and h being 9 feet, the head common to all the service deliveries:

$$H = 12{\cdot}55 + 36 - 30 - 9 = 9{\cdot}55 \text{ ft.}$$

The losses of head in this case being due to friction in FG and to the diminution of section at the point F; the general formula is therefore simply:

$$H = \frac{1}{4g}(U_5^2 + 2U_5^2) + 4\frac{L_5}{D_5}(aU_5 + bU_5^2),$$

from which proceeding as before:

$$HD_5^5 - \frac{16aqL_5}{\pi}D_5^2 - \frac{12q^2}{g\pi^2}D_5 - \frac{64bq^2L_5}{\pi^2} = 0,$$

and replacing the letters by their values, we have:

$$9{\cdot}55 \times D_5^5 - \frac{16 \times 0{\cdot}000018 \times 0{\cdot}5 \times 2040}{\pi}D_5^2$$
$$- \frac{12 \times 0{\cdot}25}{g\pi^2}D_5 - \frac{64 \times 0{\cdot}000106 \times 0{\cdot}25 \times 2040}{\pi^2} = 0,$$

or: $$D_5^5 = 0{\cdot}00975 \times D_5^2 + 0{\cdot}00099 \times D_5 + 0{\cdot}0368,$$

from which by the method of successive approximations we may quickly deduce:

$$D_5 = 0{\cdot}525 \text{ ft.},$$

and the speed will be:

$$U_5 = \frac{4 \times \cdot 5}{\pi \times \cdot 525^2} = 2 \cdot 29 \text{ ft. per sec.}$$

The losses of head in the length FG will be:

$$K_1 = \frac{4 \times 2040}{0 \cdot 525}(a \times 2 \cdot 29 = b \times 2 \cdot 29^2) = 9 \cdot 25 \text{ ft.}$$

$$K_2 = \frac{2 \cdot 29^2}{4g} \qquad = 0 \cdot 04 \text{ ft.}$$

$$\text{Total loss of head} \qquad \underline{\underline{9 \cdot 29 \text{ ft.},}}$$

which is of course: $\quad H - \frac{u^2}{2g}$ again.

Now 0·525 ft. = 6·32 inches, which is of course not a standard diameter, however by making part 7 and the remainder 6 inches in diameter we may obtain exactly the same result; the fraction of the length required of smaller diameter being calculated in the manner already explained.

A more rough and ready method of settling the ratio of the lengths of the two sizes, is to make the cost of the whole the same as it would be if the whole were of the one diameter 6·32 inches, and this were purchasable at the same rate as the standard sizes.

Thus, the price of a given length of conduit being practically proportional to the diameter, if we let x represent the length of 7 inches diameter, so that $2040 - x$ represents the length of the 6 inch diameter, we shall have:

$$7x + 6 \times (2040 - x) = 6 \cdot 32 \times 2040,$$

from which we easily deduce:

$$(7 - 6)\, x = 6 \cdot 32 \times 2040 - 6 \times 2040,$$

or:

$$x = 0 \cdot 32 \times 2040 = 653 \text{ ft.},$$

and consequently FG would be 7 inches in diameter for 653 ft. of its length, and 6 inches for the remaining 1387 ft.

A greater loss of head than allowed for in the calculation will however always be the result of adopting this method.

The diameters of only two branches C*c* and D*d* remain to be calculated.

As for the first, water flows in it in virtue of the difference in pressure between the points C where the head is N_1 and the end *c*.

Now N_1 is the difference in altitude between C and the level in the reservoir A less the losses occurring in the length AC; these losses according to earlier calculations are:

$$K_1 = 2{\cdot}38 \text{ and } K_2 = \frac{1{\cdot}415^2}{4g} = 0{\cdot}0156.$$

Hence:

$$N_1 = 60 - 39 - 2{\cdot}38 - 0{\cdot}0156 = 18{\cdot}6 \text{ say.}$$

The difference in altitude between C and *c* is:

$$Z_1 - z_1 = 39 - 36 = 3 \text{ ft.}$$

Finally as a head of 9 feet must be reserved for the head h_1 at the outlet, the effective available head will be:

$$H = N_1 + (Z_1 - z_1) - h_1 = 18{\cdot}6 + 3 - 9 = 12{\cdot}6 \text{ feet.}$$

This then is the value which comes into the ordinary formula, which latter becomes after making preliminary calculations as before:

$$Hd_1^5 - \frac{16aq}{\pi} l_1 d_1^2 - \frac{32q^2}{g\pi^2} d_1 - \frac{64bq^2 l_1}{\pi^2} = 0,$$

from which on replacing the letters by the corresponding values we may deduce:

$$d_1^4 = 0{\cdot}01143 d_1 + 0{\cdot}002 + \frac{0{\cdot}427}{d_1},$$

and then by the method of successive approximations we shall obtain:

$$d_1 = 0{\cdot}545 \text{ ft.} = 6{\cdot}55 \text{ inches.}$$

Here again part will be taken 7 and the remainder 6 inches in diameter.

The diameter of the branch D*d* may be calculated in similar fashion.

$$K_1 = 2{\cdot}38 + 2{\cdot}38 = 4{\cdot}76 \text{ ft.},$$

$$K_2 = \frac{1{\cdot}415^2 + 1{\cdot}625^2}{4g} = 0{\cdot}0363,$$

so that:

$$K = K_1 + K_2 = 4{\cdot}8 \text{ say.}$$

Therefore:

$$N_2 = Z_0 - Z_2 - 4{\cdot}8 = 60 - 42 - 4{\cdot}8 = 13{\cdot}2,$$

and:

$$Z_2 - z_2 = 42 - 39 = 3,$$

hence:

$$H = N_2 + (Z_2 - z_2) - h_2 = 13{\cdot}2 + 3 - 9 = 7{\cdot}2.$$

We may now put these values in the general equation which may be written:

$$H = \frac{2u_2^2}{g} + 4\frac{l_2}{d_2}(au_2 + bu_2^2).$$

From which by successive approximations:

$$d_2 = 0{\cdot}623 \text{ feet or } 7{\cdot}5 \text{ inches};$$

so that the pipe should be partly 8 inches in diameter and partly 7 inches, the fraction of each being calculated as explained further back.

The foregoing calculations enable the pressure heads to be determined at the beginning and end of every section of the whole system.

These pressure heads, which are given by the vertical height of a column of water borne at the point considered, would be equal to the difference in height between the point considered and the water level in the reservoir were it not for the loss of head due to the flow between these two places.

It must also be borne in mind that the free surface level falls abruptly at the junctions of lengths of different diameter, and consequently there will be two distinct heads fairly close together the one above and the other below the junction.

For example we have at the beginning of the length BC, at the point B:

$$N = 60 - 51 - \frac{U_1^2}{4g} = 60 - 51 - \frac{1{\cdot}415^2}{4g} = 8{\cdot}984.$$

At the end C the head will be increased by the difference in altitude and will be reduced by the loss due to friction, hence:

$$N_1 = 8{\cdot}984 + (51 - 42) - K_1 = 17{\cdot}98 - 2{\cdot}38 = 15{\cdot}6 \text{ ft.}$$

But immediately beyond, at the commencement of the length CD, a loss of head is produced by sudden contraction, and:

$$n_1 = 15{\cdot}6 - \frac{U_2^2}{4g} = 15{\cdot}6 - 0{\cdot}02 = 15{\cdot}58 \text{ ft.},$$

and so on.

In the example chosen, the head, and consequently the pressure, is everywhere greater than atmospheric; but it would have been otherwise, if any point in the system had been higher than the line joining the free surface levels. In the neighbourhood of such points the conduit would have acted as a siphon, and any air dissolved in the water might have collected there, checking the flow; it is advisable to avoid such defects by arranging a suitable path for the whole course of the pipes.

15. Distribution of pressure over a pipe supplied by two reservoirs. Let us now consider the following problem:

A straight pipe is fed by two reservoirs A and B, the levels in which are maintained constant. Water is taken from some

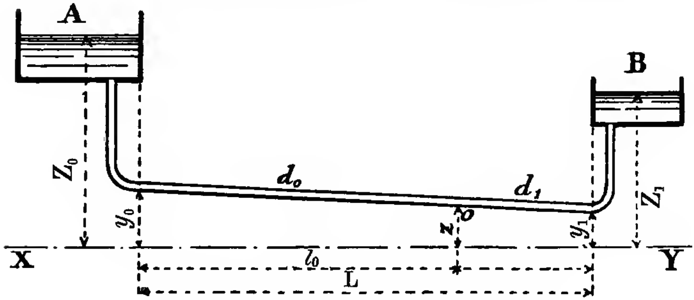

Fig. 10. Pipe supplied by two reservoirs.

point O whose position requires to be so determined that for a definite pressure there, the ratio of the quantities delivered by the two reservoirs is some particular desired fraction.

As is seen from the figure (fig. 10), the heights of the

two reservoirs above the plane of comparison XY are different, and if there were no discharge from O, the reservoir A would evidently empty itself into the reservoir B.

Obviously if, as assumed, the levels remain constant, the loss of pressure in friction to the left of O, is greater than that to the right of the same point. If a pressure column were erected at O, the water would rise in it until it reached a steady level, the height of the column being h say. The difference in height between the top of this column and the surface level in either reservoir, is the head required to generate the velocity of the water flowing from that particular reservoir plus the head absorbed in frictional losses between that same reservoir and the point of delivery O.

Let us, for the sake of simplicity, suppose that the ends of the pipes are suitably joined to the reservoirs so that only quite negligible losses of head are produced at these junctions.

We can now write the two following relations:

$$Z_0 = Z + \frac{u_0^2}{2g} + \frac{4l_0}{d_0}(au_0 + bu_0^2) + h,$$

or:

$$h = Z_0 - Z - \frac{u_0^2}{2g} - \frac{4l_0}{d_0}(au_0 + bu_0^2),$$

and:

$$h = Z_1 - Z - \frac{u_1^2}{2g} - \frac{4l_1}{d_1}(au_1 + bu_1^2),$$

and since in these two equations, we have two expressions each of whose values is equal to h, these two values must be equal to one another. Therefore:

$$Z_0 - Z - \frac{u_0^2}{2g} - \frac{4l_0}{d_0}(au_0 + bu_0^2) = Z_1 - Z - \frac{u_1^2}{2g} - \frac{4l_1}{d_1}(au_1 + bu_1^2).$$

Now the two terms $\frac{u_0^2}{2g}$ and $\frac{u_1^2}{2g}$ have, as a general rule, very small values as we have seen, and they may therefore be considered negligible, also Z may be eliminated since it appears once on each side of the equation; hence we may write:

$$Z_0 - Z_1 - \frac{4l_0}{d_0}(au_0 + bu_0^2) + \frac{4l_1}{d_1}(au_1 + bu_1^2) = 0.$$

Now: $$l_1 = L - l_0,$$

where L is the total length (given) of the conduit, and l_1 and l_0 are the lengths to the right and left of O respectively.

Moreover, the velocity of the current being inversely proportional to the sectional area, and consequently to the square of the diameter, we should have, if the quantities from the two reservoirs were equal:

$$\frac{u_1}{u_0} = \frac{d_0^2}{d_1^2},$$

or: $$u_1 = u_0 \frac{d_0^2}{d_1^2},$$

where u_1 and u_0 are the velocities in the two pipes and d_1 and d_0 the corresponding given diameters.

If the quantities instead of being equal, are in any definite ratio, say $\frac{1}{2}$ or $\frac{1}{4}$, the speed u_1 would only be $\frac{1}{2}$ or $\frac{1}{4}$ of the preceding value, that is to say the value of u_1 must be multiplied by the ratio of Q_1 to Q_0.

If in general this ratio is K, namely:

$$\frac{Q_1}{Q_0} = K,$$

then: $$u_1 = K u_0 \frac{d_0^2}{d_1^2}.$$

And as the ratios K and $\frac{d_0^2}{d_1^2}$ are given, we might for simplicity write:

$$V = K \frac{d_0^2}{d_1^2},$$

from which finally: $$u_1 = V \times u_0.$$

On replacing l_1 and u_1 by the values thus determined, in the foregoing equation, we have:

$$Z_0 - Z_1 - \frac{4l_0}{d_0}(au_0 + bu_0^2) + 4\frac{L - l_0}{d_1} V (au_0 + bVu_0^2).$$

This formula still contains two unknown quantities l_0 and u_0, but the value of h being fixed, we have a second relation

between these two quantities, which is in fact the first equation written in connection with this problem; again neglecting the relatively insignificant term $\frac{u_0^2}{2g}$ this equation may be written:

$$\frac{4l_0}{d_0}(au_0 + bu_0^2) = Z_0 - h - Z,$$

the value of Z is not given, but it is easily obtained in terms of known quantities and l_0. Let P represent the vertical fall in the conduit per foot of length, this is equal to the difference in height of the ends of the pipe divided by the total length of the latter, thus:

$$P = \frac{y_0 - y_1}{L}.$$

The altitude of the point O will be:

$$Z = y_0 - \frac{y_0 - y_1}{L} l_0 = y_0 - Pl_0.$$

Using this value in the preceding expression, we have after a certain amount of reducing:

$$l_0 = \frac{d_0 \times (Z_0 - h - y_0)}{4(au_0 + bu_0^2) - Pd_0} = \frac{M}{4(au_0 + bu_0^2) - Pd_0},$$

where for simplicity M is written for $d_0 \times (z_0 - h - y_0)$ which is a fully known quantity.

If now l_0 is replaced by this expression in the general formula, this latter only contains the one unknown quantity u_0, which can then be calculated by the method of successive approximations.

It will be seen, however, that such a substitution would give rise to a very complicated formula and a very laborious calculation.

To make this calculation somewhat easier, we might first neglect the term au_0 in comparison with bu_0^2, which as we have seen is quite legitimate when the velocity u_0 exceeds 3 feet per second (§ 9).

The general formula simplified in this manner becomes:

$$Z_0 - Z_1 - \frac{4l_0}{d_0} bu_0^2 - \frac{4l_0}{d_1} V^2 bu_0^2 + \frac{4L}{d_1} V^2 bu_0^2 = 0,$$

or:

$$Z_0 - Z_1 - 4bu_0^2 \left(\frac{1}{d_0} + \frac{V^2}{d_1}\right) l_0 + \frac{4L}{d_1} V^2 bu_0^2 = 0.$$

The expression for l_0 similarly simplified is:

$$l_0 = \frac{M}{4bu_0^2 - Pd_0},$$

therefore:

$$4bu_0^2 \left(\frac{1}{d_0} + \frac{V^2}{d_1}\right) \frac{M}{4bu_0^2 - Pd_0} = Z_0 - Z_1 + \frac{4L}{d_1} V^2 bu_0^2.$$

Reducing to common denominator, and putting all terms and factors except u_0^2 on the second side of the equation, we have:

$$u_0^2 = \frac{(4bu_0^2 - Pd_0) \times d_0 \times [(Z_0 - Z_1) d_1 + 4LV^2 bu_0^2]}{4bM (d_1 + d_0 V^2)}.$$

*By way of numerical example, let us assume the distance L between the two reservoirs is 15000 feet, and let us adopt for the other given quantities the values:

$d_0 = 2{\cdot}25$ ft. $\quad d_1 = 1{\cdot}5$ ft.
$Z_0 = 120$ ft. $\quad Z_1 = 105$ ft.
$y_0 = 45$ ft. $\quad y_1 = 36$ ft.

Let us also suppose that the head under which the delivery at O must take place is $h = 30$ feet, and finally that the ratio of the quantities supplied from the two reservoirs is:

$$\frac{Q_1}{Q} = K = \tfrac{1}{2},$$

so that twice as much water flows per second from the reservoir A as flows from the reservoir B (fig. 10).

* The example in the original, which is of course in French units, is from the work by M. Vigreux.

The value of V in the general formula may now be calculated immediately:

$$V = K\frac{d_0^2}{d_1^2} = \frac{1}{2}\left(\frac{2\cdot25}{1\cdot5}\right)^2 = 1\cdot125.$$

We have also:

$$P = \frac{y_0 - y_1}{L} = \frac{45 - 36}{15{,}000} = 0\cdot0006.$$

And again:

$$M = d_0(Z_0 - h - y_0) = 2\cdot25\,(120 - 30 - 45) = 101\cdot25.$$

Hence adopting for b the value according to Dupuit, $0\cdot0001172$:

$$u_0^2 = \frac{2\cdot25\,(0\cdot000469u_0^2 - 0\cdot00135)(22\cdot5 + 8\cdot88u_0^2)}{0\cdot000469 \times 101\cdot25\,(4\cdot342)},$$

or: $0\cdot0911u_0^2 = 0\cdot01053u_0^2 - 0\cdot012u_0^2 - 0\cdot0304 + 0\cdot00416u_0^4,$

or again: $0\cdot00416u_0^4 = 0\cdot0926u_0^2 + 0\cdot0304,$

or: $u_0^4 = 22\cdot2u^2 + 7\cdot3,$

which is now in a form suitable for solving by the method of successive approximations.

It will be found that in this particular example, several successive approximations are necessary before the final result is reached, unless a fairly near value is hit upon at first, however it may be seen by inspection that u_0^2 is just over the value $22\cdot2$, for this would be its value if the term $7\cdot3$ were absent, and we might therefore try using the value $22\cdot2$ for u_0^2 as a first attempt.

It will be found that $22\cdot5$ is the final value for u_0^2, or

$$u_0 = 4\cdot75 \text{ ft. per second.}$$

It follows that the value of u_1 in the length of diameter d_1 is according to the formula already established:

$$u_1 = Ku_0\frac{d_0^2}{d_1^2} = \frac{4\cdot75}{2}\left(\frac{2\cdot25}{1\cdot5}\right)^2 = 5\cdot35 \text{ ft. per second.}$$

From this we may deduce the quantities supplied by the two reservoirs; thus from A we have:

$$Q_0 = \frac{\pi d_0^2}{4} \times u_0 = \frac{\pi(2\frac{1}{4})^2}{4} \times 4\cdot75 = 18\cdot9 \text{ cubic feet per sec.,}$$

and from reservoir B:

$$Q_1 = \frac{\pi d_1^2}{4} \times u_1 = \frac{\pi (1\frac{1}{2})^2}{4} \times 5{\cdot}35 = 9{\cdot}45 \text{ cubic feet per sec.}$$

Total delivery at O = 28·35 cubic feet.

And for the ratio of the two quantities:

$$\frac{Q_1}{Q_0} = \frac{9{\cdot}45}{18{\cdot}9} = \tfrac{1}{2},$$

which confirms the results.

The position of the point O can now be determined; thus by the formula established at the commencement of this problem:

$$l_0 = \frac{M}{4(bu_0^2) - Pd_0} = \frac{101{\cdot}25}{0{\cdot}000469 \times 4{\cdot}75^2 - 0{\cdot}0006 \times 2{\cdot}25} = 10930 \text{ ft.}$$

By way of confirmation, the length l_1 might be calculated by an analogous formula:

$$l_1 = \frac{M_1}{4bu_1^2 - Pd_1} = 4680 \text{ ft.},$$

putting as before:

$$P = \frac{y_0 - y_1}{L}, \text{ and } M_1 = d_1(Z_1 - h - y_1).$$

It is interesting to consider certain special cases in which the formulae are very much simplified.

Suppose the diameters d_0 and d_1 are equal, as are also the deliveries, so that:

$$d_0 = d_1 \text{ and } \frac{Q_1}{Q_0} = K = 1.$$

Then the value of V which figures in the formula will be 1.

$$V = K\frac{d_0^2}{d_1^2} = 1, \text{ also } V^2 = 1.$$

Whence the general formula becomes:

$$u_0^2 = \frac{(4bu_0^2 - Pd_0)[(Z_0 - Z_1)d_0 + 4Lbu_0^2]}{8bM}.$$

This might be solved by successive approximations as before.

Finally let us assume in addition that the levels in the two reservoirs are at the same altitude and that the conduit is horizontal; these facts may be expressed by the equations:

$$Z_0 = Z_1 \text{ and } y_0 = y_1,$$

which reduce the values of $(Z_0 - Z_1)$ and of $P = \frac{y_0 - y_1}{L}$ to zero.

The general formula is therefore further simplified, and becomes:

$$u_0^2 = \frac{4bu_0^2 \times 4Lbu_0^2}{8bM},$$

from which, after making all reductions, may be deduced:

$$u_0 = \sqrt{\frac{M}{2bL}}.$$

As a numerical example, let $L = 12000$ ft., $d_0 = d_1 = 1{\cdot}5$ ft., $h = 30$ ft., $Z_0 = Z_1 = 120$ ft., and $y_0 = y_1 = 30$ ft.

Then $M = d(z - h - y) = 1{\cdot}5\,(120 - 30 - 30) = 90$,

and hence:

$$u_0 = \sqrt{\frac{90}{2 \times {\cdot}0001172 \times 12000}} = \sqrt{\frac{90}{2{\cdot}818}} = 5{\cdot}65 \text{ ft. per second.}$$

This result is sufficiently near the exact value which would be 5·85, the difference arising from the fact that the terms au have been neglected for the sake of simplicity.

If it is desired to obtain a more exact value, the initial formulae might be used in this last particular case, without neglecting the terms in au. A more complex formula will thus be obtained, but it may be solved easily by successive approximations, starting with the value of u_0 calculated by the simpler formula.

Assuming the value of u_0 to be 5·85 feet per second, it will be found that:

$$l_0 = l_1 = \frac{L}{2} = 6000 \text{ feet},$$

and:

$$Q_0 + Q_1 = 2Q_0 = \frac{2\pi d_0^2}{4} \times u_0 = 20{\cdot}65 \text{ cubic feet per second.}$$

CHAPTER III.

FLOW OF LIQUIDS IN OPEN CANALS.

16. General Law of Motion. Darcy's Formula. Instead of using cast-iron pipes or conduits of cement, concrete or other materials, in which the flowing water fills the whole cross-section, water is often taken from a stream to hydraulic works, or to the fields it is desired to irrigate, through open canals constructed according to the most favourable slopes of the soil.

Under the action of gravity the water flows in these channels with constant velocity, and since the rate of falling is not increased according to the laws of gravity, it is evident that opposing forces exist which at every instant balance the force due to gravity and counteract its action.

In fact, just as in pipes, the stream-lines sliding over the sides of the canal are subject to frictional forces, which being exerted in the direction opposite to the motion, tend to check it and compensate the acceleration due to gravity.

Not only is the water in contact with the sides affected by friction, but also the different parts of the mass of liquid exert mutual forces on one another, giving rise to a sort of viscosity which all liquids possess to a greater or less extent.

As a consequence, the stream-lines nearest the sides of the canal are retarded most, and this retardation decreases from one stream-line to the next the farther one travels from the sides, until at the centre of the stream the greatest velocity is attained.

If friction did not exist, the water would flow with uniform speed in a horizontal channel; but in consequence of the existence of resisting forces, a fall is necessary, depending on the slope of the canal, to maintain the constant velocity of flow.

Suppose there is a force F per foot of length of the stream, which is just the necessary force required to maintain the flow through one foot considered. The total force for the whole course of the canal of length L will then be $F \times L$.

This total force must counteract and therefore be equal to the frictional forces which oppose the motion of the liquid. It must be evident that the aforesaid force is proportional to the length of the canal on the one hand, to the perimeter of the section of the sides on the other hand, and also to a function of the mean speed u analogous to that which must be considered in the case of flow in pipes. Hence the frictional forces will be expressed as before by the formula $L \times P \times (a_1 u + b_1 u^2)$ where L is the length of the canal in feet, P is the *wet perimeter*, u is the mean speed of the stream-lines, and a_1 and b_1 are coefficients determined by experiment.

The force producing the motion of the water is then equal to the opposing frictional forces, and the following relation, like that which applies to pipes, will be obtained,

$$F \times L = L \times P \times (a_1 u + b_1 u^2).$$

The force F being necessarily due to the fall of the canal per foot, and the column of the flowing water being the same as that which would fill a pipe of the same section, we may regard the various cross-sections as being acted upon in virtue of the hydrostatic pressure which is exerted on them, and which constitutes the force F itself per foot of length; hence this pressure can be expressed as a function of the weight of the liquid by the formula:

$$F = S \times d \times i,$$

S being the transverse section of the canal bounded by the wet perimeter and the free surface of the liquid, d the weight

of unit volume of the liquid, and i the fall per foot of length of the stream, in other words the difference in level between two sections one foot apart, which is the slope both of the bed and of the free surface of the stream.

The relation between the opposing forces may be expressed in the form:

$$F = P \times (a_1 u + b_1 u^2),$$

and dividing each side of the equation by the factors S and d:

$$\frac{F}{S \times d} = \frac{P}{S} \times \frac{a_1 u + b_1 u^2}{d}.$$

But $\frac{F}{S \times d} = i$ as we have seen above.

Also we may write:

$$\frac{a_1 u + b_1 u^2}{d} = au + bu^2,$$

putting for simplicity:

$$\frac{a_1}{d} = a \quad \text{and} \quad \frac{b_1}{d} = b.$$

Finally, if we liken the section to a rectangle having for base the development of the wet perimeter P, and for height an appropriate dimension R, we have as a result:

$$S = P \times R \quad \text{or} \quad R = \frac{S}{P},$$

and this relation will fix exactly the value of R which is known as the *hydraulic mean depth.*

The connecting formula becomes in consequence, taking into account these last expressions:

$$i = \frac{P}{P \times R} \times (au + bu^2),$$

or:

$$R \times i = (au + bu^2).$$

This is the general formula used in solving the various problems relating to the motion of waters in open canals.

As actually the speeds of the different stream-lines vary from one stream-line to another, it is the mean speed which

must be taken into account, this is obtained by the known relation:

$$u = \frac{Q}{S}.$$

Prony has ascertained that the values of the coefficients a and b are as follows:

$$a = 0{\cdot}000044,$$

$$b = 0{\cdot}0000942.$$

Obviously, this formula, with its constant coefficients, cannot give accurate results in all cases, for it takes no account of the profile of the canal section, nor of the nature of the sides. On the contrary, the formulae due to Darcy and Bazin take these factors into consideration, and at the same time are more simply expressed as:

$$R \times i = b_1 \times u^2.$$

Vigreux states that the following are something like suitable values to give to the factor b_1 according to the nature of the sides.

Very regular sides of smooth cement or planed wood . . . . $0{\cdot}0000457\left(1 + \frac{0{\cdot}10}{R}\right)$.

Sides with well made joints, whether hewn stone, bricks or tiles . . $0{\cdot}000058\left(1 + \frac{0{\cdot}23}{R}\right)$.

Sides of ashlar work, not so well jointed $0{\cdot}0000731\left(1 + \frac{0{\cdot}82}{R}\right)$.

Mud banks . . . . . . $0{\cdot}000854\left(1 + \frac{4{\cdot}10}{R}\right)$.

These formulae, as may be seen, not only take into account the nature of the sides, but also the development of the sides in relation to the section, as they should.

We shall employ Darcy's formula when solving the different problems connected with the flow of water in open canals, such as calculations of the slope and sections of canals, and the discharge per second.

17. Gauging canals and streams. The volume of water delivered by any canal or stream depends on two factors; the transverse section and the mean velocity.

To obtain the section, the profile is determined at the two ends and at the middle of the length chosen, the area of each of these sections is calculated and their mean is taken as the section of the stream.

The velocity may be measured by divers methods, either by *floats*, by the *Woltman meter*, or by *Pitot* and *Darcy* tubes.

The floats are discs of wood about 2 inches in diameter and $\frac{1}{4}$ inch thick; they are weighted by a nail driven into the lower side of the float so that the upper surface is level with the surface of the water. These floats are dropped into mid-stream by one observer some distance up stream, two observers furnished with chronograph watches in agreement note the time at which the floats pass definite points, in the same manner that racehorses are timed at the winning-post.

The difference in the times, gives the time taken by the various floats to travel the chosen measured length of the stream; the distances traversed per second give the velocities, and the mean result of several experiments is taken.

The figure thus obtained only gives the speed of the surface of the water in mid-stream, this is different from that of the rest of the water, and appreciably higher than that of the water near the sides of the canal. If we denote the speed at the surface by V, we may suppose that the usual mean speed which enters into the calculation is:

$$U = 0{\cdot}8V,$$

in which case we have for the delivery Q across the mean section S:

$$Q = 0{\cdot}8V \times S.$$

The *Woltman meter* is a small instrument which consists essentially of a wheel with inclined vanes mounted on a horizontal axis. Suppose that this wheel is completely immersed in still water, and that it is then drawn along uniformly right through the mass of the liquid, in the

direction of its axis; the reaction of the water against the vanes will cause rotation about the axis, and the number of turns will be practically proportional to the translational velocity of the instrument; the number of turns is recorded by a clockwork train driven from the wheel spindle. The velocity of translation corresponding to each turn of the wheel per second may be determined, and this constitutes the calibration of the meter.

If now the instrument be placed in the stream whose speed is required, it is evident that the instrument being stationary the speed of rotation will be proportional to, and serve as a measure of, the desired velocity.

In practice, after having estimated the transverse profile of the chosen section, this is divided into numerous compartments so to speak, whose areas are calculated, and at the centres of which the meter is placed successively. Then the product of each of these small areas by the corresponding speed is determined, and the sum of the products gives the total delivery over the whole section.

The *Pitot tube* consists simply of a glass tube curved at right angles near one end; this is placed vertically in the current, taking care that the horizontal branch from the elbow, which is at the lower end, points up stream.

The moving water striking against the narrow orifice of this branch causes the water to rise in the vertical tube to a height which is greater, the greater the velocity of the current. The height of the column is a measure of the energy in the water, and is proportional to the square of the velocity. Thus if the water rises to a height of one inch for a particular speed, it will rise to a height of four inches for double the speed.

In reality the instrument needs to be carefully calibrated; this is done by experimentally obtaining the connection between the various heights of the column and the corresponding velocities.

In the Pitot tube, the water exerts a pressure in the horizontal branch turned up stream; if however this branch were

pointed down stream, a kind of suction or negative pressure would be produced in front of the orifice, which would cause the liquid in the vertical tube to descend below the surface of the stream.

If two such tubes are combined so as to form an inverted U tube, with the bend of the U above the water, and the two right-angled branches pointing in opposite directions immersed in the water, we have the Darcy tube; in addition a tap is mounted on the bend of the U in order that air may be drawn off and a partial vacuum produced above the columns of water in the two arms.

Water rises then in one tube and sinks in the other with respect to the level of the surface of the stream; once the instrument is calibrated, the difference in heights of the two columns gives the velocity of the current. By drawing off air from the bend, the surfaces of the columns are raised to that part of the scale which is most convenient to be read, without altering in any wise the difference in the levels, since the pressure on the two free surfaces is always equal.

We have supposed that the mean velocity of a stream is about 0·8 of the surface velocity in mid-stream; this relation varies in different rivers; for instance, for the Seine the value of the fraction is 0·62, and for the Rhine, 0·88.

We may suppose further that the mean speed is the mean speed in mid-stream at a point half-way between the water on the surface and that on the bed of the river.

In canals, the form and dimensions of the transverse section do not appear to have any great influence on the ratio $\frac{U}{V}$. Experiments have been carried out on canals of rectangular and trapezoidal section, of different depths and with both earth and masonry banks; the fraction $\frac{U}{V}$ varied from 0·80 to 0·85. The velocity of the stream-lines near the bottom and sides is less than the mean speed in proportions varying from 5 to 20 °/ₒ, according to the more or less rough condition of the banks.

In the construction of canals it is essential that the longitudinal profile should be so determined, that the velocity of flow near the sides and bottom will not exceed a certain limit compatible with the preservation of the latter. For mud banks, the velocity should be less than 4 inches per second, it varies from 0·5 to 2 feet per second for soft clays, sand, gravel and boulders; it may be as high as 4 or 5 feet per second with hewn stone and concrete, and 10 feet per second for hard rocks.

18. Problems relating to open canals. There are several problems connected with the flow of water in open canals.

It may be desired:

(1) To determine the discharge per second from a canal already constructed, of which consequently the shape, dimensions and slope supposed constant over the length considered, are known;

(2) To calculate the slope of the canal bed when the form, dimensions and delivery are known;

(3) To determine the mean speed when the other factors are known.

Fig. 11. Rectangular canal.

In order to solve these problems we shall make use of the Darcy's formula already established (§ 16):

$$R \times i = b_1 \times u^2.$$

In the first problem the discharge is calculated from the formula:

$$Q = S \times u,$$

in which S is the transverse section of the canal, and u the

speed deduced from Darcy's formula, which may obviously be written:

$$u^2 = \frac{R \times i}{b_1}.$$

Let us first suppose a canal of rectangular section, of width l and depth h; the wet perimeter will be (fig. 11):

$$P = l + 2h.$$

And the area of cross-section:

$$S = l \times h.$$

From which the hydraulic mean depth is:

$$R = \frac{S}{P} = \frac{l \times h}{l + 2h}.$$

For example, say, $l = 3$ ft., $h = 1 \cdot 5$ ft., and $i = 0 \cdot 001$ ft.

Then:

$$R = \frac{3 \times 1 \cdot 5}{3 + 2 \times 1 \cdot 15} = 0 \cdot 75; \text{ and } S = 4 \cdot 5 \text{ ft.}$$

Further b_1 may be obtained from the table of Darcy's coefficients already given; for a channel with sides of planed wood for instance:

$$b_1 = 0 \cdot 0000457 \left(1 + \frac{0 \cdot 10}{0 \cdot 75}\right) = 0 \cdot 0000519.$$

Hence: $$u^2 = \frac{0 \cdot 75 \times 0 \cdot 001}{0 \cdot 0000519} = 14 \cdot 47.$$

And: $$u = \sqrt{14 \cdot 47} = 3 \cdot 8 \text{ ft. per second.}$$

So that the discharge is obtained as:

$$Q = S \times u = 4 \cdot 5 \times 3 \cdot 8 = 17 \cdot 1 \text{ cu. ft., or } 107 \text{ gallons per second.}$$

Let us now consider the case in which the canal section is a symmetrical trapezium whose sides are inclined at 45° to the horizontal (fig. 12).

Here: $$dk = l + 2h; \quad S = (l + h) \times h,$$

applying the rule giving the area of a trapezium.

Also: $$P = l + 2bk = l + 2\sqrt{2}h,$$

since bk is the hypotenuse of the right-angled triangle bck.

If instead of the particular angle of 45 degrees, the banks are inclined at any angle whatever corresponding to a slope p,

Fig. 12. Trapezoidal canal.

then obviously bm will be equal to the depth divided by the slope and:

$$d_1k_1 = l + \frac{2h}{p}.$$

In the second problem the delivery and the cross-section being known, to calculate the required slope, the Darcy formula should be put in the form:

$$i = \frac{b_1 \times u^2}{R}.$$

Suppose that the canal of rectangular section already considered is to give a delivery of 27 cubic feet per second, we have for the mean speed:

$$u = \frac{Q}{S} = \frac{27}{4{\cdot}5} = 6 \text{ ft. per second,}$$

whence: $$i = \frac{0{\cdot}0000519 \times 36}{0{\cdot}75} = 0{\cdot}00249.$$

In the third problem we are required to calculate the speed, and use therefore the formula:

$$u^2 = \frac{R \times i}{b_1}.$$

For a slope of 1 in 400 ($i = 0{\cdot}0025$ ft. per foot of length), we have for the same rectangular canal:

$$u^2 = \frac{0{\cdot}75 \times 0{\cdot}0025}{0{\cdot}0000519} = 36{\cdot}2 \text{ and } u = 6 \text{ ft. per second.}$$

From which the delivery is:

$$Q = S \times u = 4{\cdot}5 \times 6 = 27 \text{ cubic ft. per second.}$$

If instead of a canal of rectangular section, one has to deal with a river whose depth varies irregularly over the transverse section, the sectional area cannot be exactly calculated. The breadth is divided into a number of equal parts and the depth of water measured at the dividing points, in this way the cross-section is split up into trapeziums whose areas are easily estimated, and from which the wet perimeter is obtained.

When the depth of a river is very small compared with its width, this width may be taken as the wet perimeter without appreciable error, and the formulae are simplified in consequence.

The foregoing problems relate to canals already constructed; it is evident that the same formulae will enable us to determine the sections and slopes of channels which are required to have a definite flow, with suitable speeds and slopes. The calculations would be the same whether S, i or u are unknown, according to what is given. When S is determined the dimensions of the transverse section may be settled according to the required shape.

19. Laws of Varying Movement. Standing Wave. Up to the present we have considered that the movement in the canal was uniform; this can only be so when the transverse section of the canal is the same at all points in its length, and when the slope is likewise invariable.

But this is not always the case, and it is easily seen that the speed of the water will vary from one section to another, increasing as the section diminishes, and vice versa.

In the case of uniform movement, the free surface of the water is parallel to the longitudinal profile of the stream bed, but in the case now under consideration this is not so; the surface of the water may vary from one section to another, either rising or falling according to the nature of the variations in transverse section and slope.

In consequence of such variations at consecutive points in the course of the canal, the difference in level of two neighbouring cross-sections is no longer solely dependent on the slope of the bed between the two sections, it may be less or greater according to circumstances.

In the case of varying movement then, there are actually two slopes to be considered in the neighbourhood of each section, that of the bed on the one hand, and that of the surface on the other hand, the two slopes being capable of differing very considerably, the latter either falling or rising in relation to the former according to circumstances.

Let us consider more particularly the simplest case of a bed of rectangular section whose breadth is great compared with the depth.

Here the wet perimeter is practically the breadth of the canal l, and the section and flow at the point considered are:

$$S = l \times h; \quad Q = u \times lh,$$

whence:

$$u = \frac{Q}{lh} = \frac{q}{h},$$

where q represents the quotient $\frac{Q}{l}$, the volume passing per second per foot of breadth.

As i is always used to denote the slope, and since this varies from one point to another throughout the length, we shall understand by i the slope of the bed close to the section considered, between two points situated one on either side of this section (fig. 13).

Let h be a very small increase in the depth of the water as one passes from one point to another at a very small distance d from the first; the quotient $\frac{h}{d}$ will represent the increase of depth of the water per unit length in this section, and consequently the slope of the surface in relation to that of the stream bed.

From what has been said above, it is evident that this slope of the surface depends on the slope i of the bed in the

neighbourhood considered Bb, also on the depth H at the starting point B, and the volume per foot of breadth per second. The expression for this slope is*:

$$\frac{h}{d}=\frac{i-\frac{b_1q^2}{H^3}}{1-\frac{q^2}{gH^3}}.$$

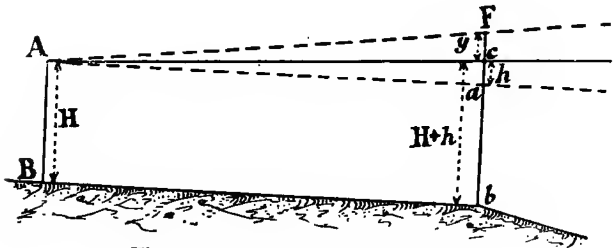

Fig. 13. Slope of surface of water.

In this formula g represents the acceleration due to gravity, namely 32 feet per second per second.

* NOTE BY TRANSLATOR. This formula is easily obtained as follows: Slope of surface required to overcome friction

$$=\frac{b_1u^2}{H}=\frac{b_1}{H}\times\frac{q^2}{H^2}=\frac{b_1q^2}{H^3}.$$

Change of level due to change of kinetic energy is:

$$\frac{u_1^2-u_2^2}{2g}=\frac{\frac{q^2}{H^2}-\frac{q^2}{(H+h)^2}}{2g}=\frac{(H+h)^2-H^2}{2gH^2(H+h)^2}q^2=\frac{H^2+2Hh+h^2-H^2}{2gH^2(H+h)^2}q^2=\frac{hq^2}{gH^3},$$

neglecting h in comparison with H.

Total difference of level $id-h$ equals then:

$$\frac{b_1q^2}{H^3}d-\frac{hq^2}{gH^3},$$

whence:

$$h\left(1-\frac{q^2}{gh^3}\right)=\left(i-\frac{b_1q^2}{H^3}\right)d,$$

or:

$$\frac{h}{d}=\frac{i-\frac{b_1q^2}{H^3}}{1-\frac{q^2}{gH^3}}.$$

Let us see what are the requisite conditions for h to be zero, that is to say for the depth of water to remain constant between two points very near together so that the surface is parallel to the bed of the channel.

In order that this should be the case the numerator of the right-hand expression should be zero, or:

$$i = \frac{b_1 q^2}{H^3},$$

whence:

$$H^3 = \frac{b_1 q^2}{i}.$$

Thus for this particular value of H corresponding to the flow q and the slope of the bed in the section considered, there will be no increase in depth between neighbouring points, and hence the surface will be parallel to the bed and the motion will be uniform.

On the other hand it may be required to discover under what circumstances d will be zero for a certain rise in level h; to this phenomenon is given the name *standing wave*, for if a sudden rise in the surface is produced at a point, the slope is so to speak infinite, that is to say the water rises vertically.

Again the formula shows the requisite conditions for this standing wave; the equation may be written thus:

$$d = \frac{1 - \dfrac{q^2}{gH^3}}{i - \dfrac{b_1 q^2}{H^3}} \times h,$$

and d will be zero if:

$$1 - \frac{q^2}{gH^3} = 0, \text{ or } g \times H^3 = q^2;$$

whence:

$$H^3 = \frac{q^2}{g}.$$

To take a numerical example, say $H = 3$ feet, so that $H^3 = 27$, then:

$$\frac{q^2}{g} = 27, \text{ or } q^2 = 27 \times 32 = 864,$$

from which q = about 29·4 cubic feet per second, per foot of width of the section under consideration, in order that a standing wave may be produced at this point.

To distinguish the two special values of H, let H_1 stand for the depth necessary for parallel and uniform flow, and H_2 be that necessary for a vertical rise of the surface, then:

$$H_1^3 = \frac{b_1 q^2}{i}, \text{ and } H_2^3 = \frac{q^2}{g},$$

and the difference:

$$H_1^3 - H_2^3 = q^2\left(\frac{b_1}{i} - \frac{1}{g}\right) = \frac{q^2}{i}\left(b_1 - \frac{i}{g}\right).$$

If when this is worked out $H_1^3 - H_2^3$ is positive, so that $H_1 - H_2$ is of course also positive, H_1 is greater than H_2 and conversely if this is negative.

As moreover this depends upon the difference $\left(b_1 - \frac{i}{g}\right)$, it is sufficient to see what is the sign of this last expression, that is to say whether b_1 is greater than $\frac{i}{g}$.

Taking for b_1 the value 0·000122 given by Tadini, and noting that if b_1 is greater than $\frac{i}{g}$, then $b_1 g$ is greater than i, or in figures:

$$0{\cdot}000122 \times 32 = 0{\cdot}0039 \text{ greater than } i,$$

or i is less than 0·004 in round numbers.

Therefore for all streams of feeble flow, that is to say those that are not torrential, H_1 will be greater than H_2, so that the depth of water for which the movement is uniform over a definite course is greater than that which gives a standing wave; on the other hand, for torrents where i is greater than 0·004, the depth of water for the standing wave is greater than that which determines uniform movement.

20. Determination of the longitudinal profile of the surface of a stream. We have examined the particular case of instantaneous change of level produced between two sections infinitely near to each other.

The more general problem is to determine the change of level occurring between two sections bounding a definite course, and more or less distant from each other.

It may be desired to know, for example, the difference in the levels of the surfaces at sections *ma* and *qd*: this change of level is represented by *qf* in figure 14.

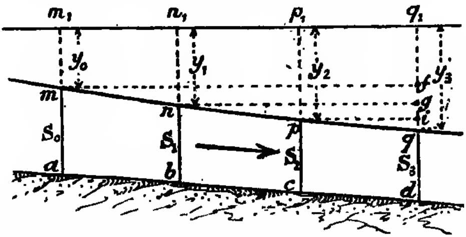

Fig. 14. Longitudinal profile of surface of water.

Since we are now dealing with an irregular stream whose transverse sections vary constantly from one point to another, as does also the slope of the bed, calculations enabling us to determine the change of level cannot be applied to the whole course, and it is necessary to limit the application to the interval between two sections which are so close together that the areas of the sections, and the slope of the bed, may be considered practically constant throughout this interval.

For this reason the course under consideration is divided into a certain number of lengths such as *ab*, *bc*, in each of which the change of level is determined separately; by doing this for one section at a time the total change of level *qf* is obtained, for it is evident that this latter is equal to the sum of the partial changes of level in the successive lengths.

In the same way also the determination of the longitudinal profile *mnpq* is arrived at between the given two extreme sections of the stream.

In order to calculate the partial changes of level which give the aforesaid profile, ordinates y_0, y_1, y_2, y_3 are sought which give the vertical distances between different points of the profile and some horizontal line such as m_1q_1.

The successive changes of level are then equal to :

$$y_1 - y_0,\ y_2 - y_1,\ \text{and}\ y_3 - y_2.$$

Also : $$qf = y_3 - y_0.$$

Considering the particular change of level :

$$qi = y_3 - y_2.$$

This represents the necessary fall for the velocity of the water to increase from u_2 to the value u_3, which values it takes successively in the sections S_2 and S_3, and also it includes the fall requisite to overcome the friction against the sides in the length *cd*.

Hence we may write :

$$y_3 - y_2 = \frac{u_3^2}{2g} - \frac{u_2^2}{2g} + \frac{P}{S} \times b_1 u^2 d,$$

in which formula P represents the wet perimeter of a mean section S, and *d* is the distance between the two sections considered.

And since : $$u_2 = \frac{Q}{S_2},\ u_3 = \frac{Q}{S_3},\ \text{and}\ u = \frac{Q}{S},$$

the foregoing equation may be written :

$$y_3 - y_2 = \frac{Q^2}{2g}\left(\frac{1}{S_3^2} - \frac{1}{S_2^2}\right) + Q^2 . \frac{P}{S^3} . b_1 d.$$

The sections *pc* and *qd* being sufficiently near to each other, the intermediate sections in this interval *d* may be supposed to vary in a very nearly uniform manner from *c* to *d*, and it is as if the channel had a uniform section S throughout the interval almost equal to the mean of the two extremes ; consequently we may suppose :

$$\frac{P}{S^3} = \frac{1}{2}\left(\frac{P_2}{S_2^3} + \frac{P_3}{S_3^3}\right).$$

Similarly, the value of b_1, which according to Darcy depends on the hydraulic mean depth, and consequently on the wet perimeter and the area of the section, may be assumed to be the mean of the two coefficients b_2 and b_3 relating to the sections considered ; so that :

$$b_1 = \tfrac{1}{2}(b_2 + b_3).$$

On replacing $\frac{P}{S^3}$ by its value, the preceding expression becomes :

$$y_3 - y_2 = \frac{Q^2}{2g}\left(\frac{1}{S_3^2} - \frac{1}{S_2^2}\right) + \frac{Q^2 \times b_1}{2}\left(\frac{P_2}{S_2^3} + \frac{P_3}{S_3^3}\right) \times d,$$

in which formula b_1 has been given the value indicated above.

In this question, Q is known, generally being determined by direct gauging; moreover, the longitudinal profile of the bed is known together with the transverse profiles at the dividing points *abcd*, and the depth of water at one end section, *am* for example; from these it is necessary to calculate y_3 so as to obain the difference in height of the surface of the water at the two extremities of the course considered.

This problem often occurs when it is necessary to construct a dam to cause a waterfall at, say, *qd*. This dam causes a choking back of the water for some distance up stream; the condition may be laid down that this choking action shall not affect the level beyond a section *am* above which an installation is situated, whose fall would be influenced unfavourably; in other words it is essential that y_0 the ordinate of the aforementioned section shall be unaltered after the erection of the dam. This height is known then, and must be preserved at a constant value which enters as a number into the equation.

To begin the calculation for the two sections *am* and *bn*, we may write :

$$y_1 - y_0 = \frac{Q^2}{2g}\left(\frac{1}{S_1^2} - \frac{1}{S_0^2}\right) + \frac{Q^2 b_1}{2}\left(\frac{P_0}{S_0^3} + \frac{P_1}{S_1^3}\right) d.$$

The value of y_1 is deduced immediately from this expression, in which all the other quantities would be known if one knew the dimensions of the section *bn*; but this latter will be modified necessarily by the erection of the dam, and therefore we must proceed by successive approximations.

To this end, assume provisionally that $y_1 = y_0$, a hypothesis which reduces the first member to zero; in which case :

$$\frac{Q^2}{2}\left(\frac{1}{g \cdot S_1^2} + \frac{b_1 P_1}{S_1^3} d\right) = \frac{Q^2}{2}\left(\frac{1}{g \cdot S_0^2} - \frac{b_1 P_0}{S_0^3} d\right),$$

let K be the numerical value of the part of the second member in brackets, the expression thus simplified becomes:

$$\frac{1}{gS_1^2} + \frac{b_1P_1}{S_1^3}d = K.$$

Take first for b_1 and P_1 the values corresponding to the section S_0, so that K_1 is known in the expression:

$$K_1 = b_1P_1d.$$

The preceding formula now becomes:

$$\frac{1}{gS_1^2} + \frac{K_1}{S_1^3} = K.$$

And reducing to a common denominator and simplifying, we have:

$$K.g.S_1^3 = S_1 + K_1g,$$

which may be written:

$$S_1^2 = \frac{1}{Kg} + \frac{K_1}{KS_1}.$$

Giving first to S_1 the value S_0 in the second member and deducing therefrom a first approximation for S_1:

$$S_1 = \sqrt{\frac{1}{Kg} + \frac{K_1}{KS_1}},$$

then recommencing the calculation using this value of S_1 in the second member, we obtain a second and closer value for S_1 by the same formula; finally after a few such operations two values of S_1 will be obtained which do not differ appreciably, and may be taken as the value sought.

When this value is inserted in the principal expression for $(y_1 - y_0)$ it enables us to calculate this difference and consequently the value of y_1.

This approximate value of y_1 serves to calculate P_1 from the equations:

$$S_1 = l \times y_1; \quad l = \frac{S_1}{y_1},$$

and:

$$P_1 = l + 2y.$$

Using this value of P_1 all the foregoing calculations could

be recommenced to obtain a new value of S_1. The result being again put in the principal equation, a second and closer value is deduced of $(y_1 - y_0)$ and therefore of y_1, this process is repeated until two successive results agree.

At the same time that values of P_1 are calculated corresponding to those of S_1 the resulting values of b_1 may be used in the formula, always remembering that b_1 must be the average of the values referring to sections S_0 and S_1.

The same procedure is followed for the succeeding intervals, and the last equation for $(y_3 - y_2)$ which has been established already, will give the value of y_3, that is to say, the height to which the crest of the dam may be built so as not to produce at *am* any rise in surface level, or any unfavourable backwash at the installation placed just up stream.

21. Lowering of Stream Surface produced by Drowned Dams. We have studied the phenomena which arise as the result of water flowing in channels of irregular profile and slope, but in our reasoning we have assumed that these variations were gradual, the change taking place almost insensibly from one point to another.

Nevertheless very sudden variations may occur, as when the banks sharply recede from or approach each other at one or more places in a stream's course, or when there is a marked depression or elevation in the bed of the stream; in addition to natural causes, there may be artificial ones, such as the construction of a drowned dam or the erection of bridge piers, which necessarily raise the bed of the stream, or restrict its transverse section.

Sudden diminution of the section of flow produces contractions which are followed by expansions, and give rise to eddies causing more or less considerable losses of head and energy. The same result follows from sudden enlargement of the transverse section.

It is evident that the preceding theory will not apply to sudden changes of section and of slope which are too irregular to be subject to calculation, but the study of the particular

case of a drowned dam or the presence of bridge piers offers no difficulty.

Let us first examine the effect of a drowned dam on the part of the stream near it, both above and below.

A dam is said to be drowned when the surface level down stream is higher than the crest of the dam. This crest being supposed sufficiently wide for the water to flow in a horizontal sheet of thickness e, composed of parallel stream-lines, we may compare this case to that of the dam considered previously (§ 7).

We there saw that on leaving a certain section at M say, above the dam, the surface of the water curved downwards, so that the surface of the sheet on the crest was lower than the level up stream (fig. 15).

If we use V to denote the velocity of the stream-lines over the dam under these conditions, and let h be the difference $(Z - e)$ which gives the height of the up stream level above the surface of the sheet close to the dam, also using z to represent the additional fall corresponding to the velocity V_0 of the water in the section nM, the following relation is now known to hold:

$$V = \sqrt{2g \times (h + z)}.$$

And since according to what has just been stated:

$$h = Z - e, \text{ and } z = \frac{V_0^2}{2g},$$

the expression giving the value of V may be put in the following form, using these values of h and z:

$$V = \sqrt{2g\left(Z + \frac{V_0^2}{2g} - e\right)}.$$

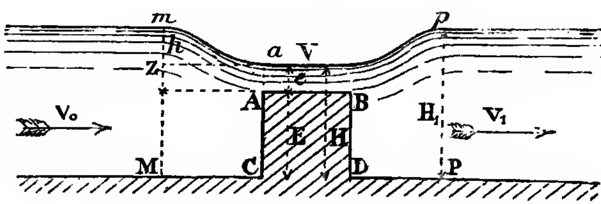

Fig. 15. Drowned dam.

Beyond the dam the volume of water expands so to speak, and the surface rises until at some section pP the stream-lines may be considered to have become parallel again.

In order to take into account all the circumstances accompanying this phenomenon, which is analogous to the standing wave already dealt with, we shall make use of the same method as when considering the loss of head due to sudden enlargement in a pipe (§ 8).

Therefore it is necessary to apply the principle of increase of momentum in the mass of water comprised between the two sections aA and pP.

The quantity which passes the section Aa per second has a momentum:

$$\frac{q \times d}{g} \times \mathrm{V} = \frac{e \times l \times \mathrm{V} \times d}{g} \times \mathrm{V} = \frac{e \times l \times d \times \mathrm{V}^2}{g},$$

where l is the length of the dam.

By the time this quantity has reached the section pP it will have acquired a new momentum expressed by:

$$\frac{\mathrm{H}_1 \times l \times \mathrm{V}_1 \times d}{g} \times \mathrm{V}_1 = \frac{\mathrm{H}_1 \times l \times d \times \mathrm{V}_1^2}{g}.$$

Hence the increase of momentum between these two sections is:

$$\frac{\mathrm{H}_1 \times l \times d \times \mathrm{V}_1^2}{g} - \frac{e \times l \times d \times \mathrm{V}^2}{g}.$$

As the motion is horizontal, the change of momentum is due to forces acting in this direction exclusively, that is to say, to the forces of hydrostatic pressure which act at the centres of gravity of the two extreme sections, say at their mid-height.

The pressure at the centre of aC exerted from left to right, in the direction of flow, is given by:

$$d \times \mathrm{H} \times l \times \frac{\mathrm{H}}{2} = \frac{d \times l \times \mathrm{H}^2}{2}.$$

That on the centre of pP has for its expression:

$$d \times \mathrm{H}_1 \times l \times \frac{\mathrm{H}_1}{2} = \frac{d \times l \times \mathrm{H}_1^2}{2}.$$

Since these two pressures act in opposite directions, the effective resultant is equal to the difference:

$$\frac{d \times l \times H^2}{2} - \frac{d \times l \times H_1^2}{2}.$$

It is now only necessary to write that the increase of momentum is equal to the above resultant due to the impulses of the two acting forces; therefore we have the equation:

$$\frac{e \times V^2}{g} - \frac{H_1 \times V_1^2}{g} = \tfrac{1}{2}(H_1^2 - H^2),$$

on eliminating the factors l and d, putting the coefficient $\frac{1}{2}$ as a common factor in the second member, and changing the signs.

The amount flowing past the two sections being the same, we have obviously:

$$e \times V = H_1 \times V_1,$$

on cancelling the common factor l, which expression may be written:

$$V_1 = \frac{e}{H_1} \times V.$$

If V_1 is replaced by this value in the foregoing relation, the latter becomes, after making all reductions:

$$\frac{2e \times V^2}{g} \times \left(1 - \frac{e}{H_1}\right) = H_1^2 - H^2.$$

From this equation the value of H_1 is deduced, and also the height of the counter-slope $H_1 - H$, when the quantities V, e, Z and H are known.

On the other hand if H_1 were known, the other quantities might be calculated with the aid of this same equation and that established at the commencement of this problem, which may be put in the form:

$$\frac{V^2}{2g} = Z + \frac{V_0^2}{2g} - e.$$

The following relations would also be used:

$$H = E + e \quad \text{and} \quad q = e \times V,$$

in which q signifies the volume of flow per second per foot of length of the dam.

The value of e is generally less than $\frac{H_1}{2}$; under these circumstances it may be stated that the height of the counter slope is less than the difference of heads corresponding to the velocities V and V_1, that is to the quantity $\frac{V^2}{2g} - \frac{V_1^2}{2g}$.

For this difference is not only used to raise the level to p, but is partially absorbed in the loss of energy due to eddies and whirls caused by the stationary wave.

Further, the value of H_1 may be greater or less than that which results from the equation established above, for it must also depend on circumstances affecting the down-stream level in each particular case.

22. Eddies produced by bridge piers. D'Aubuisson's formula. The construction of a bridge across a river, necessarily reduces the section of the current on account of the piers erected in the bed of the river. The section up stream is unaffected, it is restricted under the arch of the bridge, and then suddenly enlarged again on emerging.

The eddies which are produced below the bridge do not produce any appreciable rise in the level of the surface, and there is practically no counter-slope; these eddies are maintained for a greater or less distance after leaving the bridge according as to whether the river is at low water or in flood; in either case the normal state of flow in parallel stream-lines is re-established at a certain distance below the bridge.

Obviously the velocity of the water under the bridge will be greater than that higher up the stream, because the section offered is smaller. Not only so, but the piers and abutments guide the stream-lines obliquely towards the centre of the arch, and this tends to reduce the effective section of the stream still further.

Let us use h to represent the depth of water just below the bridge, h_1 to represent the depth just above, l the width

of the opening of the arch, L the breadth of the river above and below the bridge, and finally let d be the width of the bridge.

The mean speed of the current under the bridge close to the down-stream side, where the depth is h, is obtained from the volume of flow by:

$$Q = l \times h \times u,$$

from which:

$$u = \frac{Q}{l \times h}.$$

In consequence of friction against the sides of the structure, a loss of head is produced in the length d underneath the arch; this loss is expressed by (§ 9):

$$\text{Loss of head} = d \times \frac{\text{perimeter}}{\text{section}} \times b_1 u^2,$$

and on replacing with the letters chosen:

$$\text{Loss of head} = d \times \frac{l + 2h}{l \times h} \times b_1 u^2.$$

Now this loss corresponds to a reduction in the level which is equal to the difference of the depths $(h_1 - h)$ and we may write:

$$h_1 - h = d \times \frac{l + 2h}{l \times h} \times b_1 u^2.$$

This equation enables the difference of the depths to be calculated, and consequently the depth h_1 at the section up stream, h, that down stream being known.

This is the difference in depth necessary to overcome frictional resistance in passing the bridge, but there must also be an additional fall in surface level in order that the velocity may acquire the necessary increase in the reduced section of the structure.

If we use u_0 to represent the velocity of the stream at such a distance up the stream that it is uninfluenced by the presence of the piers and abutments, and u_1 to represent the velocity in the most contracted section of the current under the bridge; the loss of head between the two sections cor-

responding to the velocities u_0 and u_1 will be expressed by the equation:

$$Z = \frac{u_1^2}{2g} - \frac{u_0^2}{2g}.$$

Let S_0 and S_1 represent respectively the sections up stream and at the contracted part of the current under the arch, Q being the volume of water passing per second, we have:

$$Q = S_0 \times u_0 = S_1 \times u_1.$$

Whence we deduce:

$$u_0 = \frac{Q}{S_0} \quad \text{and} \quad u_1 = \frac{Q}{S_1}.$$

Putting these values in the foregoing expression, it becomes:

$$Z = \frac{Q^2}{2g \times S_1^2} - \frac{Q^2}{2g \times S_0^2} = \frac{Q^2}{2g}\left(\frac{1}{S_1^2} - \frac{1}{S_0^2}\right).$$

Since the section of flow on entering the arch is equal to $l \times h_1$, the contracted section S_1 will be equal to this multiplied by the contraction coefficient K, and we have:

$$S_1 = K \times l \times h_1.$$

As for the section S_0, it is that of the river bed above the bridge, its breadth is L by hypothesis and its depth is h_1 increased by the fall Z above mentioned; its value is then:

$$S_0 = L \times (h_1 + Z).$$

Replacing S_0 and S_1 by these values in the expression for Z, it becomes:

$$Z = \frac{Q^2}{2g}\left(\frac{1}{K^2 \times l^2 \times h_1^2} - \frac{1}{L^2 \times (h_1 + Z)^2}\right).$$

But the difference of the depths of water on entering and on leaving the arch, being due solely to the loss of head by friction against the sides of the structure, is always slight, and h_1 may be replaced by h without sensible error; hence the above expression may finally be written:

$$Z = \frac{Q^2}{2g}\left(\frac{1}{K^2 \times l^2 \times h^2} - \frac{1}{L^2 \times (h + Z)^2}\right).$$

This is d'Aubuisson's formula, and enables us, as seen, to

calculate the change of level or surface fall which is due to the erection of a structure in the bed of a river.

As regards the coefficient of contraction, K is taken as 0·85 when the piers or the abutments of the bridge present plane surfaces perpendicular to the current, and $K = 0{\cdot}95$ when the piers have their faces pointing up stream tapered to a narrow edge, forming sharp quoins which direct the water entering the arch in a more favourable manner.

It will be noticed that the unknown quantity Z appears on both sides of the formula; therefore, if arithmetical methods are adhered to, it can only be obtained from this expression by the method of successive approximations.

Z is first put equal to zero in the second member, which permits a first approximation for Z to be calculated from the formula which becomes:

$$Z = \frac{Q^2}{2g}\left(\frac{1}{K^2 \times l^2 \times h^2} - \frac{1}{L^2 \times h^2}\right).$$

This first numerical value which we will denote by Z_1, being used in the second member of the complete equation, enables us to obtain a second approximation of Z, writing:

$$Z = \frac{Q^2}{2g} \times \left(\frac{1}{K^2 \times l^2 \times h^2} - \frac{1}{L^2 \times (h + Z_1)^2}\right),$$

and so on until the value of Z thus calculated does not differ from the one found immediately before, by an appreciable quantity.

23. Pressure exerted by liquids in motion. Liquids at rest contained in a reservoir, exert on the sides of the latter normal forces which are due to their weight and which constitute *hydrostatic pressure.*

If however we consider a liquid moving past a containing wall, one of two things may occur: the fluid may either flow in a direction parallel to the side, or in a direction more or less inclined to it.

In the first case, it is evident that the pressure on the side is simply due to the weight, and equals the hydrostatic

pressure, that is to say it equals the pressure that would be exerted if the liquid were at rest.

This is no longer the case when the solid side is inclined to the direction of flow of the stream-lines. The fluid, possessed of a certain velocity, strikes the side which interferes with its direction and tends to obstruct its motion; a sort of permanent shock is produced, which is equivalent to an additional pressure, called the *live pressure* in contradistinction to the hydrostatic pressure of rest.

At this point it is advisable to call to mind the laws which govern shock between bodies. When bodies are very elastic, they experience at the instant of shock temporary deformation only, and immediately recover their original shape; then they separate and continue to move, preserving the same relative velocity as at first, but in the inverse sense, that is to say the body of smaller mass rebounds from that of larger mass, and the two bodies fly apart after the shock.

On the other hand, soft bodies, that is those possessing no elasticity, when brought suddenly into contact, continue to move together after the shock with a common velocity.

The momentum possessed by a moving body is the product of the mass of the body and its velocity.

As an example, imagine two inelastic bodies whose masses have any value whatever denoted by m and m', and let v and v' be the velocities possessed by them respectively; their moments will be $m \times v$ and $m' \times v'$ respectively. If the mass m overtakes the mass m' and the momentum $m \times v$ is greater than $m' \times v'$, the mass m' will be carried along after the shock in the direction of the mass m and the two bodies move on, keeping together and retaining the deformation occasioned by the impact, with a new velocity u different from either v or v'.

In fact after the impact, the two bodies form one mass of value $(m+m')$ and possessing a velocity u; and the momentum of the whole is equal to $(m+m') \times u$.

Now it so happens that when impact takes place between two soft bodies, the momentum of the two together is pre-

served and maintains the same value after the shock; and hence we may write the equation:

$$m \times v + m' \times v' = (m + m') \times u,$$

and dividing each side of the equation by $(m + m')$, which does not affect the equality:

$$u = \frac{mv + m'v'}{m + m'}.$$

Moreover, it may be stated that when shock takes place between soft bodies they become warmed, and the heat thus generated represents a certain quantity of work which is dissipated by radiation into the atmosphere and corresponds to a definite loss of energy.

Now the work or kinetic energy stored in a moving mass is expressed by the product $\frac{m \times v^2}{2}$.

The total kinetic energy of the system before the shock is equal to the sum of the kinetic energies of the two masses, that is $\frac{m \times v^2}{2} + \frac{m' \times v'^2}{2}$; after the shock it is reduced to the kinetic energy of the combination of the two masses, namely to $\frac{(m + m') \times u^2}{2}$.

This quantity, which represents the kinetic energy remaining stored in the combined mass, differs from the total energy possessed by the two bodies before, by all the energy corresponding to the work of deformation of the two bodies and transformed into heat as a result.

The loss of energy due to shock is then expressed by the difference:

$$\text{Loss of energy} = \frac{mv^2 + m'v'^2}{2} - \frac{(m + m') \times u^2}{2}.$$

This loss of energy has no influence whatever on the principle of the conservation of momentum; the momentum is always the same and such that the velocity u, which figures in the above expression, is simply that deduced already from the principle mentioned.

By putting this value of u in the last formula, we get, after making all reductions, in the case in which one of the masses is sufficiently small in comparison with the other as to be neglected in the calculation:

$$\text{Loss of work} = \frac{m \times (v - v')^2}{2}.$$

This is a formula which may hold when water falls on the solid portion of some machine.

It is interesting to examine the total pressure, live pressure and hydrostatic pressure which result from the impact of a jet of liquid on a fixed surface.

In this case, the column of liquid spreads to the edges of the surface, and the stream-lines, parallel at first, change their direction so as to cover the surface with a layer more or less thick.

The value of the reaction or mutual pressure which is exerted between the liquid mass and the plane, evidently depends on the inclination of the latter to the vertical, and also on the magnitude of the angle at which the jet strikes the surface.

First suppose the direction of the parallel stream-lines of the column of liquid, before the deformation, to be perpendicular to the plane. The hydrostatic pressure depends solely upon the weight of the liquid mass comprised between the point where the deformation commences and the plane. If the latter were horizontal, the pressure would be equal to the weight P of this mass, but if it is inclined to the horizontal, the pressure will be reduced to the component normal to the surface.

If K denote the slope of the plane in relation to the horizontal, this component will be given by the expression: $P \times \frac{1}{\sqrt{1 + K^2}}$; with an inclination of 45 degrees, for instance, where the slope is 1 in 1, we have:

$$P \times \frac{1}{\sqrt{1 + K^2}} = \frac{P}{\sqrt{2}} = 0{\cdot}707\,P.$$

The live pressure depends on the momentum of the liquid mass in question; let d be the weight per cubic foot of the liquid (1 cu. ft. of water weighs 62·4 lbs.), let S be the section of the parallel column in square feet, and let V be the velocity with which the jet strikes the stationary plane, the weight striking the plane per second will be $d \times S \times V$, and the mass:

$$m = \frac{d \times S \times V}{g},$$

where g is the acceleration due to gravity, namely 32 feet per second per second. The momentum of the mass considered will be therefore:

$$\frac{d \times S \times V}{g} \times V = \frac{d \times S \times V^2}{g}.$$

Now it may be shown that this momentum is a measure of the live pressure on the fixed surface.

If the stream-lines make a certain angle with the normal to the surface, contrary to the original hypothesis, only the component in the direction of the normal will come into play.

Thus if K′ denote the slope of the direction of the stream-lines in relation to the perpendicular to the plane, the component required is expressed by:

$$\frac{d \times S \times V^2}{g} \times \frac{1}{\sqrt{1 + K'^2}}.$$

By adding together the two pressures, we have for the total pressure R on the surface:

$$R = \frac{P}{\sqrt{1 + K^2}} + \frac{d \times S \times V^2}{g \times \sqrt{1 + K'^2}}.$$

Let us now pass on to consider the case of a thin plate, such as AB, placed inside a cylindrical pipe, and normal to the direction of the current (fig. 16).

The current is divided by the plate, and eddies are produced both before and behind it; the stream-lines cease to be parallel after a certain section ab, and only become so again beyond the plate at some such section as cd.

The said plate being completely surrounded by the fluid,

is subject to a pressure R_0 on the side facing up the stream, and a pressure R_1 in the opposite direction on the other face; therefore the resultant pressure will be $R_2 = R_0 - R_1$, and this pressure will be exactly equal to the reaction of the plate against the liquid.

Let S be the section of the pipe, p_0 the pressure per unit surface on the up-stream side, and p_1 the pressure per unit

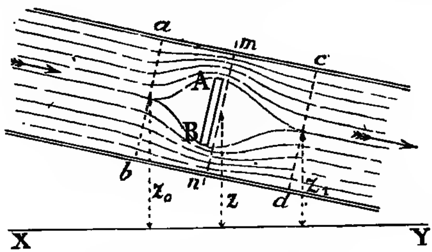

Fig. 16. Thin plate in a pipe.

area on the other, then the resultant pressure exerted in the direction of the axis of the pipe normal to the plate will be: $S \times (p_0 - p_1)$.

It is also necessary to take into account the weight of liquid which comes into the phenomenon, namely that comprised in the volume *abcd* which is the seat of eddies produced by the presence of the plate. It is always the component of this weight in the direction of the current that must be taken into account. This is obtained by projecting the vertical force which represents the weight, on the axis of the pipe, or inversely by projecting this direction on the vertical.

If d be the weight of the liquid per cubic foot, the weight of the cylindrical mass will be $d \times S \times ac$, and as the projection of ac on the vertical is equal to $Z_0 - Z_1$, the component of the weight to be considered is: $d \times S \times (Z_0 - Z_1)$.

The two forces considered above, that is the resultant of the pressures on the one hand, and the action of gravity acting through the weight of the liquid on the other hand,

must be equal to the reaction R_2 opposed by the plate to the liquid, and we may write:

$$R_2 = d \times S \times \left(Z_0 - Z_1 + \frac{p_0}{d} - \frac{p_1}{d}\right).$$

The speed V′ in the contracted section *mn*, very near to the side of the plate facing down the stream, is necessarily greater than the speed V in the full section *ab*. Using p to represent the pressure per unit area at *mn*, the application of Bernoulli's theorem (§ 2) enables us to write:

$$\frac{V'^2}{2g} - \frac{V^2}{2g} = Z_0 - Z + \frac{p_0}{d} - \frac{p}{d}.$$

At *cd* the velocity is the same as at *ab*, since the two sections of the current are equal; on the other hand, if we apply the same theorem between the sections *mn* and *cd*, it would be necessary to take account of the loss of head due to sudden enlargement, which is produced immediately beyond the plate. This loss of head is expressed by (§ 8): $\frac{(V'-V)^2}{2g}$, and hence we may put:

$$\frac{V^2 - V'^2}{2g} = Z - Z_1 + \frac{p}{d} - \frac{p_1}{d} - \frac{(V'-V)^2}{2g}.$$

One may always add equations together, member to member, or subtract them from each other; doing this for the two preceding equations, we obtain after making all reductions:

$$\frac{(V'-V)^2}{2g} = Z_0 - Z_1 + \frac{p_0}{d} - \frac{p_1}{d}.$$

And noticing that the second side of this equation is equal to that part of the expression for R_2 enclosed in brackets, we have:

$$R_2 = d \times S \times \frac{(V'-V)^2}{2g}.$$

This then is the value of the mutual reaction between the plate and the liquid.

It may easily be shown that this reaction can be put in the form:

$$R_2 = d \times Q \times \frac{V^2}{2g} \times K,$$

in which Q is the area of the plate AB, and K is a numerical coefficient which varies from 3·66 to 1·13 as the ratio $\frac{S}{Q}$ of the section of the pipe to the area of the plate itself, varies from 2 to 20.

Suppose for example $\frac{S}{Q} = 2$, $d =$ the weight of water per cubic foot $= 62{\cdot}4$ lbs., $V = 1$ foot per second, and $Q = 4$ square feet; the formula gives us:

$$R = 62{\cdot}4 \times 4 \times \frac{1}{2g} \times 3{\cdot}66 = 14{\cdot}25 \text{ lbs.}$$

If we replace the thin plate by a short cylinder, placed with its axis coinciding with that of the pipe, supposed horizontal, the reaction of the end of the cylinder facing up the stream, and inversely the pressure of the liquid on this end, will be given by an analogous formula:

$$R_2' = d \times Q \times \frac{V^2}{2g} \times K',$$

but the numerical coefficient K′ is smaller than K, often being less than half.

The cylinder instead of presenting a flat end to meet the current, might be furnished with a prow, so arranged that the transition from the full section, to the annular space between the cylinder and the pipe, would be more gradual; in this case K′ would be diminished because the contraction of the stream is less.

When bodies are placed in a canal whose section is very large in comparison with their own, the conditions of motion of the water are very different from those considered above, and it becomes impossible to calculate the mutual reactions.

It is natural to suppose however that the reactions may be expressed by an analogous formula:

$$R = d \times Q \times \frac{V^2}{2g} \times K,$$

in which K must be determined by experiment. This has been done by Dubuat, who has found the values of K applying to the different cases of a sheet, a cube or a prism, immersed in a current of large section to be 1·86, 1·46 and 1·34 respectively.

Instead of considering a fixed body immersed in a stream, one may consider the case of floating bodies furnished with bow and stern; the resistance experienced by these bodies, when moving through liquids, will always be expressed by the same formula.

The value of K depends on whether the water is at rest or in motion; it varies considerably with the form of the extremities, namely, the bow and stern of the floating body.

For flat surfaces, both front and rear, $K = 1·10$; with a very sharply pointed prow, $K = 1·00$; if in addition the rear is provided with a tapered stern, $K = 0·50$; finally for ships, the value of K may be as low as 0·16.

CHAPTER IV.

HYDRAULIC ENGINES.

24. Power of waterfalls. Hydraulic installations. Hydraulic motors and engines are used to receive the energy of a waterfall and transform this into mechanical work. As is always the case, this transformation is not wholly carried out, for certain losses are produced as the result of eddies and shocks between the molecules of water both on entering and inside the engine, by friction and also by reduction of velocity of the water on leaving the motor.

If P be used to denote the number of lbs. of water passing through a motor in unit time, and falling through a height H measured between two sections situated close to the motor, one above and the other below the fall, then the energy given out from this mass of water each second is, by the very definition of mechanical work, equal to $P \times H$.

Further this mass, possessing a velocity u_0, has also a definite kinetic energy:

$$\frac{m \times u_0^2}{2} = \frac{P \times u_0^2}{2g},$$

and hence the total energy stored in the water is:

$$P \times H + \frac{P \times u_0^2}{2g} = P \times \left(H + \frac{u_0^2}{2g}\right).$$

But on the other hand, the water leaving the motor still possesses a certain residual velocity u_1, which is essential in order that the water may be able to flow away by the tail-

race, and it thus carries away with it a quantity of energy equal to $\frac{P \times u_1^2}{2g}$ which is lost as far as useful work is concerned.

To be precise, the energy supplied to the motor is equal to:

$$P \times \left(H + \frac{u_0^2}{2g} - \frac{u_1^2}{2g}\right).$$

In practice the velocities u_0 and u_1 are not very different, and the two corresponding terms are therefore practically equal and cancel; hence the expression for the work or energy supplied to the motor is reduced simply to:

$$T_m = P \times H.$$

But, as already stated, this energy is not entirely transformed into useful mechanical work by the motor, because of the different sources of loss indicated, especially friction; if T_f be used to denote the total amount of work lost, the expression for the useful work T_u is:

$$T_u = T_m - T_f = P \times H - T_f.$$

Therefore the efficiency, which is by definition the ratio of the useful work to the total work supplied, will be:

$$\frac{T_u}{T_m} = \frac{P \times H - T_f}{P \times H}.$$

Thus the efficiency will be greater the smaller is T_f, the term to be subtracted; in the construction of motors, and in the arrangement of hydraulic installations, the attempt is made to secure the most favourable conditions, so as to attain the greatest possible reduction of the loss T_f.

A waterfall is produced either by a natural or an artificial barrier. The height of the fall is the difference in the levels of the surface of the water at two sections, one just above and the other just below the barrier.

The tail-race, which is the part of the water-course situated close to the dam and on the down-stream side of the works, is not affected at all from the point of view of variations in the surface level, by the presence of the dam.

But it is not the same with regard to the surface level of the water coming to the dam, this level may be considerably modified as we have seen already (§ 20).

The erection of hydraulic works on any water-course is subject to legal restrictions. These impose the construction of regulating devices, and the powers granted by the authority fix the nature and dimensions of these works, such as the altitude of the crest of the dam producing the fall.

It is necessary in fact to provide for getting rid of floods, and for this reason the dam really consists of two parts, a fixed barrier and a waste channel provided with sluice gates. As the level during ordinary floods must not cover the top of the dam, the area of the sluice gates when full open must be so calculated as to be able to deal with the necessary discharge.

When an existing installation is to be used, it is necessary to distinguish between the case in which the work is authorised, and that in which it is unauthorised. In the first case, the height of the crest of the dam being fixed by the order issued by the authority, it is not necessary to consider works which may be established on the stream just above.

As for works situated down the stream, account must be taken of the fact that the height of the crown of the dam there affects the level of water at the point under consideration; now the level of the water in the tail-race of the installation must be higher than the crest of the dam down the stream, by the full height which corresponds to the slope necessary for the flow of water between the two installations. Therefore, knowing the height of the dam in question, the actual height of the available fall may be deduced.

If the construction of the installation is unauthorised, it will be necessary to become acquainted with the conditions holding at the nearest works up the stream; in fact, the latter is assumed to require that its working shall not be affected by an unauthorised installation, and that its fall shall be in no wise diminished by the presence of the latter.

In order to find the available fall in the installation it is desired to construct, the sluice gates in the dam should be

raised to their extreme limit, so as to get rid of the fall entirely. Then either the level of the tail-race of the installation above will remain stationary, or will be lowered by an amount which will show the possible reduction in the fall available.

In the first case we could maintain the water at the level of the crest of the dam; but in the second case it would be necessary to so reduce the depth that the level in the tail-race of the works above should come down to the lower level found in the foregoing experiment.

It is also essential to consider the draining of the land and fields bordering the river between the two installations; in order that this draining may take place in a normal manner, it is necessary that everywhere throughout the course of the river, the surface of the water should be kept 6 or 8 inches below the lowest part of the fields lying on either bank.

The level of the supply being thus fixed, that down the stream will be determined by considerations set forth above, in the first case dealt with.

By the aid of dams and discharge sluices, the level of the source is maintained constant during low water and normal weather; it only rises during floods. On the other hand the level below varies continually with the variable flow of the river. Hence the difference of levels above and below, which gives the height of the fall, and consequently the energy available, fluctuates during the year.

It is advantageous therefore to know the mean condition of the water, either by numerous gaugings, or by information gathered on the spot, or preserved in Public records and meteorological reports over several consecutive years.

25. Subdivision of flow among different installations. *Undrowned orifices.* It is interesting to study the particular case in which several installations are supplied with water from the same reach or at the same fall; in this case it is necessary to apportion the hydraulic power between the different users according to the rights of each.

Let B be the common supply canal or millhead (fig. 17); let A be one of the intake canals to a corresponding installation; let O be the supply orifice built in the retaining wall or dam which determines the magnitude of the fall H to the level in the channel leading to the hydraulic motors; let N be the level at which the upper stream must be maintained practically constant either during drought or normal conditions.

If we use e to represent the height of the opening O during drought, and z to represent the depth of water above the upper edge a of the opening on the up-stream side, the sill b will be situated at a depth $(z+e)$ below the level of the supply surface.

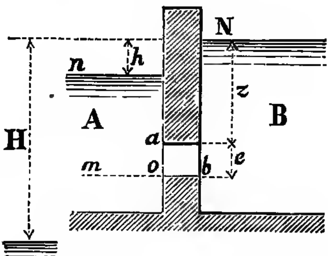

Fig. 17. Division of supply of water by water-gates.

It may be seen at once that water cannot flow from the upper stream into the intake canal through the orifice O, without a certain lowering of the surface level h in A, for it is in virtue of this difference of level h between N and n, that the flow is produced, from one side of the barrier to the other (§ 4).

It consequently follows that the lower is n, the level in the intake canal of the installation, the greater h will be and the greater will be the quantity of water flowing through the opening O. But, on the other hand, the actual available fall which is equal to $H - h$ will be decreased, and will be smaller in proportion as the change of level h is greater.

It is seen therefore that the same circumstance which tends to increase the quantity flowing, also tends to diminish the available fall; if the fall were a maximum and equal to H,

there would be no loss of level, but the quantity flowing would be reduced to zero; if on the other hand the change of level were equal to H, there would no longer be any available fall, and in each case the power would be zero. It may be perceived that there must be some intermediate loss of level corresponding to a fall and a delivery giving the maximum motive power to which each user may lay claim, according to their acquired rights.

In every case it is certain that the maximum flow is attained when the water level in the intake canal is maintained at the level of the sill *mb*. Accordingly it is evident that the position of the sill giving, with this maximum delivery, the greatest possible amount of power, will correspond to the maximum power that can be obtained.

Suppose then that $\left(z+\frac{e}{2}\right)$ is the head above the centre of gravity of the opening; the level in the intake canal being maintained at the same height as the sill *m*, the height of the available fall will be reduced to $(\mathrm{H}-z-e)$, and the mechanical energy of the fall which is proportional to the quantity flowing and to the height of the fall will be proportional to the product:

$$(\mathrm{H}-z-e) \times \sqrt{z+\frac{e}{2}}.$$

This expression might be put in another form by putting:

$$x = z + \frac{e}{2},$$

whence:

$$z = x - \frac{e}{2},$$

which makes it:

$$\left(\mathrm{H} - x - \frac{e}{2}\right) \times \sqrt{x}.$$

Now it may be shown by the calculus that this product, and consequently the work, has a maximum value when x has given to it the value:

$$x = \frac{2\mathrm{H} - e}{6}.$$

Whence putting this in the expression for z, we obtain:

$$z = \frac{H - 2e}{3},$$

and:

$$z + e = \frac{H + e}{3}.$$

The height $(z + e)$ will determine the depth of sill common to all the openings to the intake canals of the different installations, and during the season of drought the height of these openings will be e. But during heavy floods, the orifice should be increased until its upper edge is as high as the level N, which has to be maintained stationary, if one does not desire to lose water over the dam and through the sluice gates of the upper reach.

It will be necessary in every case that the width l of the orifices to each installation, of which we have not yet spoken, should be so designed as to supply to each user that part of the whole flow of the stream to which he has the right.

Let Q be the quantity flowing in times of drought, and Q′ that during flood, and similarly let q and q' be the fractions due to the installation considered; one may then write the proportion:

$$\frac{q'}{q} = \frac{Q'}{Q} = K,$$

whence:

$$q' = K \times q,$$

where K denotes the ratio of the flow at high water to that during drought.

At low water, the discharge per second through each orifice will be:

$$m \times l \times e \times \sqrt{2g \times \left(z + \frac{e}{2}\right)} = q.$$

At high water, the orifice being open to the level N acts, as already stated, like a dam; the quantity flowing will therefore be given by the expression:

$$m' \times l \times (z + e) \times \sqrt{2g \times (z + e)} = q' = K \times q.$$

In these two equations m and m' are the coefficients of

contraction corresponding to the openings e and $(z+e)$ of the orifice respectively; m' and m have for mean values 0·44 and 0·65, hence the ratio $\frac{m'}{m} = \frac{0{\cdot}44}{0{\cdot}65} = 0{\cdot}68 = \mathrm{U}$ may be considered constant; one may put then $m' = 0{\cdot}68m = \mathrm{U} \times m$.

The two unknown quantities in the equations for q and q' are e and l; e may be deduced from the first equation and its value used in the second; after all calculations and reductions have been made, it will be found that:

$$(9\mathrm{K}^2 + 2\mathrm{U}^2)e^3 + 6\mathrm{H}(\mathrm{U}^2 - 3\mathrm{K}^2)e^2 + 6\mathrm{U}^2\mathrm{H}^2 \times e + 2\mathrm{U}^2\mathrm{H}^3 = 0.$$

This equation enables e to be calculated by successive approximations.

In the particular instance where $\mathrm{K}=1$ and $\mathrm{Q}=\mathrm{Q}'$, that is when the flow of water is constant, the two preceding relations become one, since q also equals q'; now since two unknowns cannot be derived from one equation, one could make use of the first simply, and giving to e an arbitrary value, the value of l might be deduced corresponding to the desired discharge q.

Obviously the values possible to e are not unlimited; they must lie between O and $\frac{\mathrm{H}}{2}$, for e must necessarily be greater than O, but cannot be greater than $\frac{\mathrm{H}}{2}$; in fact for the latter value, we should have:

$$z = \frac{\mathrm{H} - 2e}{3} = \frac{\mathrm{H} - \mathrm{H}}{3} = 0,$$

that is to say, the level N would be lowered to the upper edge of the orifice, and for a value a trifle greater it would come below it, which is inadmissible.

Let us take as a numerical example of the general case: $\mathrm{K} = 1{\cdot}41$, whence $\mathrm{K}^2 = 2$; $\mathrm{U}^2 = 0{\cdot}45$; $\mathrm{H} = 3$; the preceding equation becomes:

$$(9 \times 2 + 2 \times 0{\cdot}45)e^3 + 6 \times 3(0{\cdot}45 - 3 \times 2)e^2$$
$$+ 6 \times 9 \times 0{\cdot}45e + 2 \times 0{\cdot}45 \times 27 = 0,$$

or: $$18{\cdot}9e^3 - 99{\cdot}9e^2 + 24{\cdot}3e + 24{\cdot}3 = 0,$$

which may be written:

$$99{\cdot}9e^2 = 18{\cdot}9e^3 + 24{\cdot}3e + 24{\cdot}3,$$

or again: $$e^2 = 0{\cdot}191e^3 + 0{\cdot}245e + 0{\cdot}245.$$

Putting at first $e = 0{\cdot}6$ feet in the second member, it will become:

$$e^2 = 0{\cdot}041 + 0{\cdot}147 + 0{\cdot}245,$$

whence: $$e = 0{\cdot}66.$$

And putting anew this value in the second member of the equation, we find that $e = 0{\cdot}67$, that is to say a value sufficiently near that assumed in the latter case as to be considered correct.

After e has thus been determined, the value of l will be deduced from either of the expressions for q and q', so that each user may be supplied, under the stated conditions, with the volume of water to which he has the right.

There will then be no danger that one of the interested parties may draw off a greater quantity of water than his share, since the same height of orifice sill has been chosen as the controlling level in each of the canals leading away; this will be the same for high water when the orifice is opened to the maximum amount, namely right up to the level N of the surface in the head-race.

For intermediate discharges between low water and high water, it is evident that all the water gates should be raised by the same amount in order that no user may be deprived of his rights.

Drowned orifices. The method of dividing the water which we have been considering, is characterised by the fact that the sill of the openings serving the different works, constitutes the regulating level for the water in these canals, so that the orifices are quite free on the down-stream side, and may be considered as never being drowned.

Just in proportion as the water gate is raised to increase the height of the orifice as measured from the sill, according

to the value of the delivery varying from a state of low water to the season of floods, so the level of the intake will be raised until it reaches the level N at the latter times, except during exceptional floods.

It is seen then that with this system of undrowned orifices, the actual fall $H - h$ will vary according to the value of h, which depends itself on the variations in the flow of the water-course; hence the interested parties are compelled to submit to a loss from this cause, which will be greater the nearer the volume of water flowing corresponds to that of flood.

It may however be proposed to so make the division that the loss of available fall shall be constant for all conditions of flow.

Everything will be arranged as before with this difference, that the level n in the intake canal, instead of being lowered to the sill *ob* of the orifice, that is the minimum possible height, is fixed at a definite height n below which the surface of the water is never allowed to descend.

To satisfy these conditions, it is obviously necessary that the level thus fixed should be such that the orifice will always be below it, that is, it must remain completely drowned.

Hence the lower edge of the gate shown as a (fig. 17) cannot be raised above the adopted level n; in this latter position the opening of the orifice will be the maximum, and we will denote its height by E. If we suppose, as is generally the case, that the sill *ob* of the orifice is level with the bottom of the intake canal, the symbol E will represent the height of the lower edge of the gate, measured from this bottom.

It is unnecessary to state that the height of the opening E is that which corresponds to the flow in time of flood; similarly let us use e to denote the minimum height of opening of the gate corresponding to times of drought.

In addition to the foregoing conditions which the level n must satisfy, n must be so chosen that in the stated conditions, the motive power of the available fall may be a maximum.

Now this power is proportional both to the height of the

actual fall $(H-h)$, and to the delivery which is proportional to the square root of the head h, it is therefore proportional to the expression :

$$(H-h)\times\sqrt{h}=H\times\sqrt{h}-\sqrt{h^3}.$$

It may be shown, moreover, that this power has a maximum value when h is made equal to $\frac{H}{3}$.

Therefore this relation settles the position of the level n, which is chosen as the height below which no user must be allowed to let the surface of the water in his particular canal descend.

Hence the lower edge of the water gate must never be raised above the level n thus obtained, and for this extreme position, it should be at a distance $\frac{H}{3}$ below the surface N.

Using q' as before to denote the maximum flow when the water is high, and q to denote the minimum flow at low water, and putting also $q'=K\times q$, the coefficient K may vary from, say, one to three.

Again, let l be the constant width of the orifices, m the contraction coefficient in the passage of water under the gate, and we shall have for the deliveries q and q', or $K\times q$:

$$q=m\times l\times e\times\sqrt{2g\times\frac{H}{3}},$$

and :

$$K\times q=m\times l\times E\times\sqrt{2g\times\frac{H}{3}}.$$

If we take the ratio of these equations, side for side, we find :

$$\frac{K\times q}{q}=K=\frac{E}{e}.$$

That is to say the deliveries are proportional to the opening of the gate, as would naturally be expected.

These three equations enable us to determine the unknown values of the three quantities l, E and e, if H, K and q are unknown.

In order to fix these ideas by a numerical example, let us give the following values to the different quantities entering the formulae, to wit: $K=2$, $m=0{\cdot}65$, $H=10$ feet, and $q=100$ cubic feet; the foregoing equations now become:

$$100=0{\cdot}65\times l\times e\times\sqrt{64\times 3{\cdot}33},$$

whence: $$100=9{\cdot}47\times l\times e;$$

further we have: $$2\times 100=9{\cdot}47\times l\times E,$$

and finally: $$2=\frac{E}{e}.$$

The three equations may finally be written:

$$l\times e=10{\cdot}57,$$
$$l\times E=21{\cdot}14,$$
$$E=2e.$$

Account must now be taken of the ratio to be observed between the depth of water in the canal and the breadth of the same, which is the same as that of the orifice; the ratio generally adopted is $\frac{E}{l}=2$ or $l=\frac{E}{2}$, this condition corresponding to the least slope per foot necessary for the flow of water in the canal, since the ratio of wet perimeter to section is then a minimum; from this, replacing l by this value in the preceding expressions, we have finally:

$$\frac{E\times e}{2}=10{\cdot}57,$$

$$\frac{E^2}{2}=21{\cdot}14.$$

From which equations E and e are easily deduced, and consequently also the width l of the orifice.

Dams. The water could be divided equally well by means of a dam. To this end, the wall which separates the common upper reach or mill-head A from the intake canal of each user is pierced with rectangular notches, for each of which the sill ab is placed at a certain height above the bottom of the canal mn (fig. 18).

Hence the water flows over the edge *ab* as over a dam, and it is evident that the discharge is as great as possible when the level N′ in the canal is in line with the crest *ab*, and that therefore no advantage will be gained by the user by allowing the surface level in his canal to descend below this height.

The problem then is simply to determine the depth *h* of the top of the dam below the surface N of the upper reach, which ought to be maintained stationary, whether in time of drought or flood, except in very exceptional cases.

This depth might be determined in two ways. Firstly, one might take as starting point the flow in flood time, and

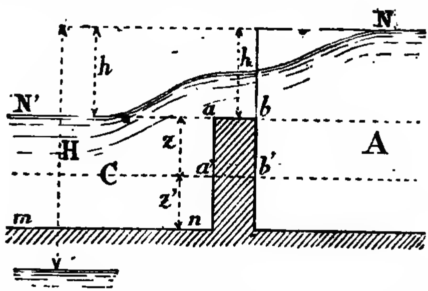

Fig. 18. Subdivision of water-course by a dam.

find what ought to be the value of *h* corresponding to this flow for maximum work to be obtained, with the greatest possible flow per foot of width of each notch. Secondly, one might fix straight away the dimension *h* to suit the flow during seasons of low water, but it would then be necessary to lower the crown of the dam to allow passage to heavy floods.

Let us in the first place take the former case. The surface level N′ being fixed for the greatest flow, must be raised as the state of low water or drought is approached, together with the crest of the dam. In fact the level N′ should always be maintained in line with the crest *ab*, whatever be the height of the latter above the bottom of the channel.

As before, it is seen that the available fall is reduced to $(H-h)$, the amount h being the fraction of the fall which must be sacrificed in order to obtain the necessary flow over the dam.

This flow is proportional to the height of the section of flow h and to the square root of this same height, which therefore makes the volume of flow proportional to $h \times \sqrt{h}$ (§ 7). The power, which is proportional to the volume of water flowing and to the height of the available fall, will then be finally proportional to the expression :

$$(H-h) \times h\sqrt{h}.$$

Hence the value of h, which makes this expression a maximum, will be the best to adopt for the condition of greatest volume of water flowing per second.

It may be shown that the said value ought to be :

$$h = \frac{3H}{5}.$$

Assuming then this value for h, we can now determine the width l of the opening corresponding to the maximum delivery due to each partaker, by a relation analogous to that used in the former cases :

$$m \times l \times h \times \sqrt{2g \times h} = K \times q,$$

q being always used to signify the quantity delivered during drought, and $K \times q$ the maximum when the water is plentiful.

In times of drought, the depth of the dam top must be reduced to the value h', such that :

$$m \times l \times h' \times \sqrt{2g \times h'} = q.$$

These various equations enable us to calculate h and h', and consequently the total amount of rise $(h-h')$ of the vertical gate. For intermediate quantities all the gates must be raised equally to heights varying between h and h'.

In the second case, the surface N′ fixed for least flow in the channel must be maintained at the same level whatever the quantity flowing, save of course during exceptional floods.

For drought then, we shall have as above:

$$m \times l \times h \times \sqrt{2g \times h} = q.$$

In flood time, the crest of the dam must be lowered to the level $a'b'$, and as N′ is to be maintained at the same height as before, the dam will now be drowned under a depth of water z.

The thickness of the sheet of water, taking account of the two coefficients of contraction m and m', will be made up of two layers $m \times h$ and $m' \times z$, such that the total volume flowing during flood time will be:

$$(m \times h \times l + m' \times z \times l) \times \sqrt{2g \times h} = \mathrm{K} \times q.$$

This equation holds on the assumption that all the stream-lines in the superposed layers have a common velocity $\sqrt{2gh}$; for this to be so, it is necessary that the lower position of the crest $a'b'$ should be at a height z' above the bottom of the canal, at least equal to h. Under these conditions, the depth of water in the intake canal will be at least $(z + h)$, and the breadth of the notch will be taken equal to that of the canal:

$$l = 2\,(z + h).$$

Putting this value in the preceding expressions, they become:

$$2m \times (z + h) \times h \times \sqrt{2gh} = q,$$

and: $$2\,(z + h) \times (m \times h + m' \times z) \times \sqrt{2gh} = \mathrm{K} \times q.$$

If we take $m = 0{\cdot}40$, and $m' = 0{\cdot}60$, the ratio $\dfrac{m}{m'}$ will be equal to $\frac{2}{3}$, and from the formulae above, after making all reductions, we might deduce the relation:

$$\frac{z}{h} = \tfrac{2}{3}\,(\mathrm{K} - 1),$$

whence: $$z = \tfrac{2}{3}\,(\mathrm{K} - 1) \times h.$$

Putting for simplicity in the general case:

$$\tfrac{2}{3}\,(\mathrm{K} - 1) = \mathrm{K}',$$

we may write: $$z = \mathrm{K}' \times h.$$

Using this value in the first equation, it becomes:

$$2m \times (K' \times h + h)\, h \times \sqrt{2gh} = q,$$

or:

$$2m \times (K' + 1) \times h^2 \times \sqrt{2gh} = q,$$

from which after all reductions are made, we obtain:

$$h^5 = \frac{q^2}{8g \times m^2 \times (K' + 1)^2} = M.$$

As all the quantities in the second member are known, it reduces to a figure which we will denote by M. From this, h can be obtained either by taking the fifth root of M with the aid of logarithms, or by successive approximations from:

$$h^4 = \frac{M}{h}.$$

Suppose for example that $M = 100$; at first putting $h = 1$ in the second member, we have:

$$h^4 = \tfrac{100}{1},$$

whence:

$$h^2 = 10 \quad \text{and} \quad h = 3{\cdot}16.$$

Now using this value of h in the second member, it becomes:

$$h^4 = \frac{100}{3{\cdot}16} = 31{\cdot}6,$$

whence:

$$h = 2{\cdot}4 \text{ feet.}$$

A third such operation would give $h = 2{\cdot}5$ feet, which is so near the preceding value that it may be taken as practically exact.

Evidently this method of dividing the waters requires those concerned to always keep a fixed level in their supply canal; but on the other hand it causes only a minimum loss of fall which is that corresponding to the condition of least flow.

Canal bed. In the last case, the sill ab of the notch in the wall separating the upper stream from the supply channel, virtually constitutes a dam.

Let us suppose that this sill is lowered to $a'b'$ for example,

and that this level forms also the bed of the intake canal (fig. 18). This canal bed will be equivalent to a dam of indefinite breadth in the direction of the current. Hence we know (§ 7) that for a suitable depth Z of water in the upper stream above the sill of the dam, that is above the bed of the intake canal, the maximum volume Q flowing is given by the equation:

$$Q = 0{\cdot}385 l \times Z \times \sqrt{2gZ}.$$

In this formula, l represents the transverse width of the canal, and Z is the depth of water as defined above. This maximum flow corresponds moreover to a lowering of the surface level by an amount h, between the upper stream and the canal, such that:

$$h = \frac{Z}{3}.$$

If as before, we use q to denote the volume which must be supplied to each of those concerned at a time of minimum flow, and K represents the ratio of maximum to minimum flow, it is evident that the depth Z of the bed of the canal for the maximum flow during flood, must be obtained for a value of Q defined by the equation:

$$Q = K \times q.$$

This value, used in the preceding equation, allows us to write:

$$K \times q = 0{\cdot}385 l \times Z \times \sqrt{2gZ}.$$

We know moreover that, for a definite flow, the intake canal may be inclined by the least possible amount, if the condition expressed in the following formula be realised:

$$Z - h = \frac{l}{2},$$

in other words,* when the breadth of the canal is double the depth of water in the same.

* Note by Translator. For a definite area the square has the least perimeter of all rectangles, and obviously therefore for an open rectangular canal the half-square will have least wet perimeter in relation to sectional area.

If these particular conditions are satisfied, it is to be noticed that:

$$Z - h = \tfrac{2}{3}Z = \frac{l}{2},$$

whence:

$$l = \tfrac{4}{3}Z,$$

and using this value of l in the principal equation, we shall have:

$$0{\cdot}385 \times \tfrac{4}{3}Z^2 \times \sqrt{2gZ} = K \times q,$$

or on reducing:

$$\frac{1{\cdot}54\sqrt{2g \times Z}}{3} \times Z^2 = K \times q.$$

We will deduce the value of Z^2 from this equation by writing:

$$Z^2 = \frac{3K \times q}{1{\cdot}54\sqrt{2gZ}}.$$

By way of numerical example, let $K \times q = 30$ cubic feet, then:

$$Z^2 = \frac{3 \times 30}{1{\cdot}54 \times 8 \times \sqrt{Z}} = \frac{7{\cdot}3}{\sqrt{Z}},$$

and from this the value of Z can be deduced by successive approximations. Assume that $Z = 1{\cdot}5$ say in the second member, so that:

$$Z^2 = \frac{7{\cdot}3}{\sqrt{1{\cdot}5}} = 5{\cdot}95 \quad \text{and} \quad Z = 2{\cdot}44.$$

On placing this value of Z in the second member of the same formula, it is found that $Z = 2{\cdot}16$, and a still closer approximation will be obtained on putting this second value anew in the formula. Say $Z = 2{\cdot}2$ is the exact value, from this it follows that the surface is lowered by an amount h given by:

$$h = \tfrac{1}{3}Z = 0{\cdot}73 \text{ ft.}$$

The depth Z of the canal bed being thus determined from the flow corresponding to time of flood, the surface lowering

corresponding to the limited flow during drought, will be expressed by:

$$m \times l \times (Z - h) \times \sqrt{2gh} = q.$$

In this formula q is the minimum flow, m is the coefficient of contraction on entering the intake channel, l is the width of this canal, and $(Z - h)$ is the depth of water in the same.

Replacing l by $\frac{4}{3}Z$ the formula becomes:

$$m \times \tfrac{4}{3}Z \times (Z - h) \times \sqrt{2gh} = q,$$

and this last equation may be put in the form:

$$h = Z - \frac{3q}{4mZ \times \sqrt{2gh}}.$$

From which as before the value of the lowering of the surface corresponding to minimum flow may be obtained by successive approximations.

It is seen that, in this system of allotment, in which the sill of each intake orifice is fixed, the flow is regulated solely by the lowering of surface level; therefore it is impossible to require that each user should maintain a constant level in his intake; it is necessary on the contrary to erect a graduated scale, giviug the different allowable levels according to the state of the water-course.

However it is not practicable to estimate each day the total quantity of water flowing, in order to assign the height of the required level; therefore this method of allotment does not satisfactorily fulfil the requirements of the case, when the act states a very definite fraction of the total flow.

If, as is often the case, the order simply specifies the height of the sill and the dimensions of the orifice for each installation, each user may maintain the surface in his own intake at the most convenient height to himself, which is that giving maximum power of waterfall.

It can be shown that the value of h corresponding to this maximum power is given by the expression:

$$h = \frac{3(H + Z) \dot{-} \sqrt{9(H + Z)^2 - 20H \times Z}}{10}.$$

Thus this value depends on the height H of the fall, and it would be necessary to make h vary according to the different values of this fall, if each party wished to obtain the full amount of power according to his rights.

Further, h could not be maintained constant, because for the same level N in the upper reach, it would be necessary to make h vary according to the variable flow of the stream, in such a manner as to divide the available volume of water in proportion to the rights of the interested parties.

In order to satisfy these various requirements, h should be calculated by the formula given above, taking for H the value of the total height of fall during drought; then by means of a vertical gate capable of being moved across the orifice, the breadth of this opening should be varied according to the flow Q of the stream.

It will be necessary moreover, in order to ensure the observation of the contract clauses, that all the gates be worked by a common mechanism, and that each receive a displacement proportional to the width of its particular opening.

Instead of dividing the energy between the interested parties, the agreement may require the apportioning of the total flow of the stream according to volume. Under these circumstances, since the maximum flow corresponds to $h = \frac{Z}{3}$, each user will be allowed to produce this reduction of level in his intake.

The maximum delivery per second for each of the canals of width l will be given by the expression:

$$Q = 0{\cdot}385 l \times Z \times \sqrt{2gZ}.$$

It will be necessary to so reduce the breadth of the openings that each supply is subject to reductions proportionate to that of the total flow of the stream.

26. Classification of hydraulic motors. We have previously explained the laws controlling the movement of water and its distribution. As we have seen, water in move-

ment is a source of energy; it now remains to be explained how this energy may be received and transformed into useful work. Therefore we come to the study of hydraulic motors.

It should be noted at the outset, that water is particularly well adapted for the production of mechanical energy, because it is first of all a fluid, that is to say a substance whose molecules are easily displaced with regard to one another, and its mass, subject to the earth's attraction, flows easily and acquires the velocity which gives it its kinetic energy, and it is easily turned into receptacles whose form it takes, and from which it escapes without difficulty.

Moreover water is a heavy body, and in consequence is capable of possessing a considerable amount of energy even when at rest, this energy being manifested by the pressure that the liquid exerts on the containing sides, and which may be transformed into effective work, if these sides are capable of movement.

We see then that water is capable of producing work in several different ways, either by giving up its kinetic energy, like a hammer which does its work of forging when its velocity is destroyed; or by hydrostatic pressure on the moving parts of a suitable mechanism, or in virtue of its kinetic energy and its pressure both acting at the same time.

These considerations show us that hydraulic motors ought to be suitably designed according to the method of using the motive power, and they can be classed, from this point of view in two categories.

Those which use at the same time both the kinetic energy and the weight of the water, are *bucket wheels*, and *breast wheels* with flat blades.

Those which derive their energy from the kinetic energy of the current are *hanging wheels*, *undershot water-wheels*, and *turbines*.

Water-wheels are essentially different from turbines, in construction, form and manner of installation, turbines being the true users of kinetic energy.

For this reason we propose to group the different sorts of

water-wheels on the one hand and turbines on the other hand, as is customary.

The former class are divided into *overshot wheels* which receive the water on the highest part of the rim; *side wheels* and *breast wheels* in which the water is introduced at the side a little above the horizontal line passing through the centre of the wheel; and finally *undershot wheels* which receive the water against the lowest paddles on the under side of the wheel.

We see from this description that the use of water-wheels is limited to falls of small height, which must practically be less than 40 feet. In fact the wheels which are best adapted for high falls being of the overshot type, we see that in this case the wheel must be nearly as high as the fall, and a wheel could not exceed 40 feet in diameter without being of an impracticable size and weight.

Water-wheels always have horizontal axes; turbines however may be mounted on either vertical or horizontal axes.

In turbines, water acts in various ways according to the kind and manner of installation. The water may completely fill the compartments of the moving wheel and exert a certain hydrostatic pressure on the inside surfaces of the channels comprised between the vanes; in this case the turbine is called a *reaction turbine.*

If the compartments of the moving crown have a section greater than that strictly corresponding to normal flow, the sheets of water do not completely fill the moving channels, they flow freely over the concave surfaces of the blades, whilst transmitting the greater part of their kinetic energy to them. This mode of action characterises the *free deviation* or *impulse turbine.*

Finally the channels may be filled completely, without the sheets of water passing through the wheel being subject to any pressure, this is the case with turbines *limited to no reaction.*

Obviously the types whose spaces are completely filled with water, work indifferently whether in air or in water,

that is to say, the turbines may be completely submerged in the tail-race, or, be *drowned*, as it is technically called.

Several advantages result from this. Firstly the turbine may be placed as low as possible, so as to utilise the maximum amount of fall. In consequence of its immersion, the turbine can coutinue to turn, even during sharp frosts, when only the surface of the stream is frozen over.

From the point of view of erection, the use of a drowned turbine allows the ground-floor of the turbine-house to be left free, and hence the floor-space occupied is nothing so to speak.

Generally speaking, turbines are as suitable for small falls as for high. But they are especially advantageous in the latter case, and they are the motors *par excellence* for the lofty natural falls found in mountainous districts. In fact, the energy being the product of the quantity of water delivered, multiplied by the pressure or the height of the fall, it is easily seen that for a definite power, the greater the height of fall, the less is the volume of water flowing, and the smaller will be the dimensions of the turbine.

Moreover water-wheels only receive the working fluid in one or two buckets at a time, whereas turbines are fed over the whole periphery; this circumstance still further reduces the relative dimensions of the latter.

It may be added that turbines always work at higher speeds than water-wheels which turn very slowly and consequently require complicated transmission gear from the wheel-axle to the working machinery. If further, it be noted that turbines are constructed entirely of metal, it will be seen that the cost of their maintenance is insignificant. Finally it is to be observed that the sense of rotation of turbines, contrary to that of wheels, is independent of the direction of the current.

On the other hand turbines have the disadvantage that they are more complicated in construction and more difficult to install than wheels; they require the bed of the canal to be more deeply excavated than do ordinary wheels, and they

are more easily damaged by foreign bodies which the water may carry over.

27. Description of the different types of water-wheels. Before describing in detail the various hydraulic motors, we will pass them in review very rapidly so as to give a general survey of them and their uses in each practical case.

Overshot bucket wheels may work either with or without a *head of water.*

The latter arrangement requires a supply canal which delivers the water on the top of the wheel, without being checked by a vertical water-gate dipping into the channel and limiting the flow. The water thus flowing freely must be in the form of a sheet of constant thickness, this system is therefore suitable when the level of the mill-head scarcely varies.

When the surface and the volume are both subject to variation, the wheel with the head of water is used. The channel then terminates with a vertical gate which allows the opening to be regulated according to the depth of water which forms a head more or less great above the orifice.

These wheels are used for falls of from 10 to 40 feet and for volumes of flowing water from 100 to at most 350 gallons. The latter wheels turn more rapidly and consequently deal with more water for the same dimensions. The question of speed is of importance when machinery, such as dynamos which run at high speeds and which require variable driving force, are to be driven. Here, the wheel which works at the higher speed will be preferable because it allows the train of gear wheels and the transmission to be simplified, and moreover it acts as a flywheel making the motion regular in spite of momentary variations of head.

When the surface level of the tail-race is variable, it is necessary to change its direction so that the vanes do not turn in the opposite direction to the current in the canal. This troublesome construction may be avoided however by

using the *backshot wheel* in which a contrivance fixed on the end of the supply channel directs the sheet of water in a backward direction making the wheel turn in the required sense.

The *breast wheel* is designed to receive the water on its side. It turns therefore in the direction of the water in the tail-race, and may be drowned to a depth of 4 or 5 inches like the preceding wheel. The water flows through a shuttle which directs the stream-lines on to the blades of the wheel.

Evidently these wheels are suitable for small falls, they are used for falls of 8 to 10 feet, and when both upper and lower surface levels are subject to variation.

In these different wheels the water escapes from the buckets before the latter have arrived at their lowest point, and thus a portion of the work due to the weight of the water is lost. To remedy this disadvantage, the lower quarter of the wheel on the supply side is encased, so to speak, in *a circular breast* which encloses the buckets and prevents the spilling of the liquid.

This breast, besides adding to the expense of the installation, also possesses the disadvantage of allowing an appreciable amount of water to pass, if the clearance between it and the wheel is too much, or if the clearance is insufficient, it gives rise to friction between the wheel and the channel.

Millot's wheel is devised to keep the water in the buckets until the lowest point is reached, without the use of a breast. To accomplish this end, very deep buckets are arranged to fit one inside another and the outside of the bucket is curved according to the exterior circumference. The buckets are consequently almost completely enclosed on the outside, and the water has to be introduced on the inner side of the buckets.

All the foregoing wheels receive the water above the centre; the *side wheel* also is made to turn in a circular channel but receiving the water below the centre.

In the case of the wheel without head of water, the supply channel terminates in a *swan's neck*, which is a

continuation of the circular breast and against which a gate is inclined, over which the water flows into the wheel.

This wheel is used for falls of from 3 to 5 feet. When the level in the millhead and the flow are variable, the supply is regulated by sluices, and the water is supplied under a head, flowing through the orifice made by raising the gate.

In these wheels the paddles are not shut in on the sides by *shrouds* carried on the spokes of the wheel, but the water is confined by a masonry channel, having two *wing-walls* or *cheeks* which are a continuation of those of the feeding canal.

The *Sagebien siphon-wheel* is another side wheel, but it is constructed on a different principle. The very deep blades are flat and are arranged obliquely and not in the direction of the radii of the wheel. The water enters the wheel by passing over the gate of the swan's neck, but not by falling as in the case of a dam. The water fills the buckets in succession, as these are immersed in the supply channel, and the surface in the spaces takes the same level as in this channel.

Thus this wheel acts as a water meter and its speed is proportional to the flow. It is useful for small falls of from 2 to 9 feet, and is capable of dealing with a great volume of water per second. But it is costly to construct on account of its large size, and it turns at a slow speed which makes it rather apt to work like a machine of variable resistance, instead of exerting a flywheel action on the axle.

Undershot wheels are characterised by the fact that the supply is admitted underneath a gate which directs the water against the lower blades of the wheel. In this case, the water does not act by its weight, but only in virtue of its kinetic energy. This wheel is generally used for falls of less than 3 feet. It is simple in construction, runs at high speed, and consequently has a large flow per foot of width.

The efficiency of the undershot wheel with *flat paddles* is less than that of the breast wheel. Poncelet has considerably improved this efficiency by replacing the flat blades

with curved ones and excavating the bed of the channel in cylindrical fashion, at the end of which excavation the curve is continued by the curve of the wheel blade. By this means the water is caused to enter the buckets without shock and the efficiency is raised from 35 per cent. to 60 per cent.

Hanging or *floating wheels* are the simplest imaginable; the blades are flat and dip into the river one after the other; these wheels work in the open river, without the water being dammed, and without gates. The motion is produced solely in virtue of the kinetic energy of the current.

As the level of the river varies, the wheel has to follow its fluctuations, for this purpose the wheel may be mounted on a boat, or as in the system devised by Colladon it may be arranged to float on the water like a barrel with projecting paddles.

Finally *American* wheels must be mentioned, notably the *Pelton* wheel which is specially constructed for high falls. In this wheel a jet of water strikes the buckets directly, and therefore this wheel works under conditions intermediate between those for ordinary wheels and turbines.

28. Description of the different kinds of Turbines. Parallel and inward-flow turbines may work either as *impulse turbines* or as *reaction turbines.*

In the first case, we know that the speed of the water at the outlet of the *guiding crown* is that due to the hydrostatic pressure at the level considered.

For this to be so, the outlet section of the *moving vanes* must necessarily be larger than that of the channels in the guiding crown; the water then flows in a medium in which the pressure is atmospheric, as in an open canal, and to accomplish this end the interior of the moving crown is put in communication with the atmosphere by means of *vents* arranged in the cheeks of the crown.

It follows from this that free deviation turbines or impulse turbines as they are called, must never be drowned. They should be installed above the tail-race level in such a manner

that the base of the moving crown clears it by some inches. Consequently this type of turbine is only suitable where the tail-race level is not liable to much variation.

When the water completely fills the moving compartments, its velocity on leaving the guide ring is less than that due to the height of the fall. Hence the water reacts on the moving blades exerting a certain pressure which is added to the kinetic energy of the liquid jets and assists in producing rotation of the moving crown.

Under these conditions, the buckets being full, the flow is constant. Such a turbine can work drowned; it is placed just above the lowest water level, and the moving crown is then more or less submerged according to the variations of tail-race level.

In the *outward-flow turbine*, of which the best known type is the Fourneyron, the fixed and moving crowns are concentric, the moving crown being the outer one. The water runs into a chamber, of which the turbine forms the base; it penetrates the guide channels and flows horizontally from the centre to the circumference.

The flow through such turbines is proportional to their speed of rotation. This fact is very unfavourable to the successful running of an installation, for an increase of torque reduces both the speed and the power, when the latter ought to be increased; conversely, the power increases when the resistance of the machinery diminishes.

The *inward-flow turbine* has an inverse arrangement, the moving crown being inside, the water flows from the circumference horizontally towards the centre. The flow diminishes as the speed of rotation is increased by the reduction of the torque, contrary to what takes place in the outward-flow type, and the motive power increases as the machinery demands a greater amount of energy.

The regulation of the inward-flow turbine is therefore to some extent automatic, this constituting a marked advantage over other systems of hydraulic motors.

In the *parallel-flow turbine* the crowns are superposed, so

that the water flows in a direction parallel to the axis. The moving crown which is underneath is level with the tail-race; it may be drowned, however, if the turbine can work as a *reaction* turbine.

An arrangement has been designed by Jonval which allows the turbine to be placed at a point intermediate between the upper and the lower level of the fall. The turbine is enclosed in a vertical tube which is curved so as to discharge into the tail-race horizontally.

With an arrangement of this sort, the turbine utilises the whole fall, whatever is its position in the inside of the tube, provided that the orifice of the pipe is always completely under water. If this condition be not fulfilled, the column of water in the pipe is broken, and the continuity of the hydrostatic pressure is lost. This device, which can be equally well applied to centrifugal and centripetal-flow turbines, goes under the name of *jonvalisation* of turbines.

The advantage of free deviation can be obtained, even when the turbines are placed under the tail-race level by using the artifice of *hydropneumatisation* due to Girard. This consists in placing the turbine under a dome which is charged with compressed air by means of a pump driven by the turbine itself, the air being at a sufficient pressure to keep the water level inside the dome below the lower face of the turbine.

With this device, the useful fall for turbines with concentric crowns is diminished by half the thickness of the moving crown; but in parallel-flow turbines, hydropneumatisation does not cause any loss of fall.

When the guide crown is less than 3 feet below the surface of the water, vortices are formed which suck in air, and this air being drawn into the guiding channels, gives rise to considerable turmoil in the movement of the jets, and a reduction in the flow and in the efficiency of the turbine is the result.

To avoid this disadvantage, Girard proposed that the water should be supplied by means of a *siphon* which would take its supply from the millhead, then rise above the surface level there, and end in an annular ring at the guide crown.

Thanks to this device, it is no longer necessary to drown the turbine when there is insufficient depth of water over the openings of the fixed crown. The total fall is always utilised, but the column of water above the turbine is increased by all the superelevation of the bend of the siphon above the millhead level.

For a very high fall with small flow, a part only of the moving crown may be supplied with advantage; in this way the speed of the turbine is reduced and the sections of the channels may be enlarged, which is a favourable condition for the avoidance of obstruction.

The fixed crown is then reduced to a *circular segment* into which a pipe of the same form delivers the water under pressure.

For several years continental makers have been engaged in copying the methods of American manufacturers, who have brought out a new type of turbine known as *mixed-flow*. These combine the inward and parallel-flow turbine in one. The water, guided by the exterior fixed crown, enters the moving compartments in a direction at right angles to the axis, then gradually takes a direction parallel to the axis in order to escape into the tail-race.

These turbines may be either reaction or impulse turbines, drowned or undrowned, and may be provided with the Jonval suction tube or subject to hydropneumatisation.

They possess the advantages of the inward-flow type from the point of view of automatic regulation, and at the same time, like the parallel-flow turbine, they can utilise the whole fall without being drowned in the tail-race.

The appearance of these turbines is very different from that of the types already mentioned. The vanes are helicoidal or spoon-shaped; they are also remarkable for their great depth, which sometimes equals the mean radius of the crowns. As a result, the outlet area is bounded by curved lines giving a large development, which allows the diameter of the turbine to be considerably reduced and secures great economy in manufacture.

The Hercules, Leffel and New-American turbines are the best known of this type.

In spite of the claims made for their turbines by American manufacturers, it is certain that the efficiencies published are exaggerated; they do not surpass, if ever they attain, the efficiencies obtained by French constructors.

Water-Wheels.

29. Overshot Bucket-Wheel without head of water. The classification and general description of hydraulic motors having been dealt with, each type will now be considered in detail.

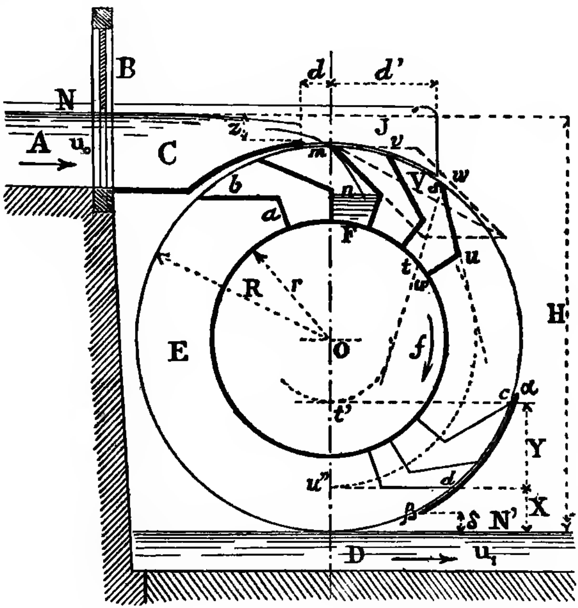

Fig. 19. Overshot wheel without head of water.

Figure 19 shows a water-wheel of the first type.

The buckets are formed by vanes or partitions made in two parts, one part a in line with the radius of the wheel, the other part b inclined in a definite direction determined in a manner to be seen presently. The bottom of the buckets is formed by the rim or *sole-plate* F; the side pieces are made by two *cheeks* or *shrouds* E. The whole is bolted to arms assembled on the *hub* and supported by the axle.

This wheel, as its name indicates, is supplied from above, and receives the sheet of water at its highest point m.

The rectangular supply canal A is prolonged by a trough C which projects over the wheel, and terminates in a spout almost concentric with the wheel. The spout ends at a distance d behind the vertical diameter Om of the wheel, and this end is at a depth z below the level N of the surface in the intake. The spout is provided with cheeks which project beyond the vertical diameter in order to constrain the sheet of water and direct it into the buckets.

The wheel works as follows: water flows into the wheel in the form of a parabolic sheet and acts in two ways: (1) by the kinetic energy which it has acquired in falling from the level N to the summit of the wheel through the height z; (2) by its weight when lying in the buckets.

The useful power of this wheel is given by the expression:

$$Tu = P \times H - Tf,$$

where P is the weight in lbs. (gallons multiplied by 10) delivered per second, H is the height between the surfaces in the tail-race and the intake respectively, and Tf is the sum of all the losses of energy that take place in the working of the wheel.

These losses are of different kinds. The first, due to friction against the sides of the spout C, is negligible when care has been taken to make the surfaces quite smooth.

In the second place, the water enters the buckets with an absolute velocity V, resulting from the height of the fall z. If the circumferential speed v of the wheel were the same, the water would enter the buckets in a state of relative rest,

so to speak, but as v is always less than V, the water enters with a certain relative velocity w. The energy corresponding to this speed is lost in eddies in the buckets; it is equivalent to a loss of fall $\frac{w^2}{2g}$, and the resulting loss of energy is:

$$tf = P \times \frac{w^2}{2g}.$$

Thirdly, the buckets do not keep their charge till they reach the lowest point of their circular course; they commence to let the water escape when they reach a point c and are quite empty at d, this is while still at a height X above the lowest point.

To find the position of the point c, it should be noticed that the buckets are not completely filled, but only partially so. The fraction filled being known, from the extremity s of any partition, $u'us$, draw a straight line st such that the capacity corresponding to the area $suu't$ is precisely equal to the volume of water with which the bucket is charged. The straight line st has now only to be drawn in the horizontal position ct', making it turn about the centre O to obtain the desired point. As for the point d, it can be determined similarly by drawing the part of the paddle us on the horizontal $u''d$.

Y represents the height of the fall during which the charge in the buckets varies from p to O. It may be assumed then that the result is approximately the same as if the charge were reduced to half, and remained constant throughout the fall Y, or as if half the fall $\frac{Y}{2}$ were lost. Finally, to this loss must be added the loss of fall X from the point d to the tail-race level.

Hence the two chief losses of energy will be expressed by:

$$tf = P \times \left(\frac{Y}{2} + X\right).$$

Finally as the water falls from the buckets between c and d, it leaves the wheel with a velocity v equal to that

of the wheel, and the resulting loss of energy has for its value:

$$tf = \mathrm{P} \times \frac{v^2}{2g}.$$

The sum of all the losses, important and otherwise, is given by:

$$tf = \mathrm{P} \times \left(\frac{w^2}{2g} + \frac{v^2}{2g} + \frac{\mathrm{Y}}{2} + \mathrm{X}\right).$$

It can be shown that if the absolute velocity V of the water flowing into the buckets is given, the sum $\frac{w^2 + v^2}{2g}$ will be a minimum when the circumferential speed v is half the projection of the absolute velocity V on the direction of v.

Referring to the parallelogram of velocities v, w and V, if V be given in magnitude and direction, the value of v is easily determined, and the third side w of the triangle of velocities follows at once.

Now the direction of w is exactly that which must be given to the outer portion of the paddle in order that the water may enter the buckets without shock, it therefore settles the line of this part of the paddle.

As a matter of fact, the blades are not restricted to this line, for as a general rule it would have the result of making the outer blade practically in line with the radius. In practice a direction making an angle of 25 or 30 degrees with v is adopted. Under these conditions, shocks against the blades are not avoided, and the velocity V being thus destroyed, an amount of energy is lost equal to the corresponding kinetic energy $\frac{\mathrm{V}^2}{2g}$. This loss will replace that referring to the relative speed in the expression for the loss of work which now becomes:

$$\mathrm{T}f = \mathrm{P} \left(\frac{\mathrm{V}^2}{2g} + \frac{v^2}{2g} + \frac{\mathrm{Y}}{2} + \mathrm{X}\right).$$

The absolute velocity V of the water pouring on to the wheel depends on the thickness z adopted for the sheet of water. This depth should lie between 4 and 8 inches; the

former is suitable for small falls, and the latter for falls and quantities of water rather greater.

The volume of water flowing from the dam is calculated, as we have already seen, from the formula:

$$Q = l \times q = 0{\cdot}385\, lz \times \sqrt{2gz},$$

in which q is the volume in cubic feet delivered per second per foot of width of the dam, z of course being expressed in feet. q varies from 0·594 to 1·71 cu. ft. (3·7 to 10·7 gallons) per second for thicknesses 4 and 8 inches respectively of the sheet of water.

The breadth l of the trough is deduced from the relation:

$$l = \frac{Q}{q}.$$

Further, the absolute velocity of the water at the end of the spout is given by:

$$V_0 = \sqrt{2g \times \frac{z}{3}},$$

and is 2·7 feet per second for $z = \frac{1}{3}$ of a foot, and 3·8 feet per second for $z = \frac{2}{3}$ of a foot.

The velocity V with which the water enters the buckets is the resultant of the horizontal velocity V_0 and the vertical velocity due to the height of fall; these two velocities form the two sides at right angles of a right-angled triangle of which the speed V is the hypotenuse, according to the rule of the triangle of velocity vectors.

The actual thickness of the sheet of water being equal to $\frac{2}{3} z$, the central stream-line leaves the reservoir at a point situated above the top of the wheel by an amount equal to $\frac{z}{3}$, increased by the thickness of the lip and the necessary clearance between it and the wheel, say, $\frac{z}{3} + 1$ inch; the velocity due to this fall is:

$$u = \sqrt{2g\left(\frac{z}{3} + \frac{1}{12}\right)},$$

whence:

$$V^2 = \sqrt{V_0^2 + 2g\left(\frac{z}{3} + \frac{1}{12}\right)} = \sqrt{2g\left(\frac{2z}{3} + \frac{1}{12}\right)},$$

since $V_0^2 = 2g \times \frac{z}{3}$.

Hence $V = 4{\cdot}43$ feet per second for $z = \frac{1}{3}$ feet, and $V = 5{\cdot}8$ feet per second for $z = \frac{2}{3}$ feet.

The connection between v and V obtained above is not satisfied in practice, for the reasons indicated above; moreover it would limit the peripheral velocity of the wheel to 2·9 feet per second, which is too low. It is generally between 3 and 5 feet in order to eliminate any unevenness in turning due to want of balance, that is, eccentricity of centre of gravity, which produces periodic variations in the speed, these variations being more appreciable the slower the speed of the wheel.

The peripheral velocity v being known, the number of turns per minute will be given by:

$$n = \frac{v \times 60}{2\pi R}.$$

The diameter 2R of the wheel is:

$$2R = H - (z + \tfrac{1}{12}).$$

The breadth l' of the wheel ought to be greater than that l of the trough to avoid loss of water by spilling at the sides, let us say:

$$l' = l + 4''.$$

As regards the inside radius r of the crown, which enables the depth $(R - r)$ of the buckets to be determined, this can be deduced from the equation:

$$r = R \times \sqrt{1 - \frac{2Q}{K \times K' \times v \times l' \times R}}$$

in which K is the fraction filled, and may be between $\frac{1}{3}$ and $\frac{2}{3}$, Q is the total volume flowing per second, and K′ is

a coefficient which allows for the thickness of the blades and lies between 0·90 and 0·95, as the thickness varies from 1 to 2 inches.

Bucket wheels without head of water are used for falls of from 10 to 40 feet.

They are only suitable for small streams, for they are scarcely capable of dealing with more than 9 gallons per second per foot of length. A flow of 180 gallons per second would therefore require a wheel 20 feet wide, which is about the maximum practicable.

Further, as the supply is over a dam, it is required that the level of the dam should be nearly constant. Therefore the flow cannot be varied appreciably, and the wheel thus arranged is not suitable for a stream with variable flow, especially when regulated by *sluice gates.*

Moreover the peripheral speed of this wheel does not often exceed 4 feet per second; under such conditions it cannot act as a flywheel, regulating the speed in spite of the variable demand of the machinery. Further, this slow speed requires the installation of complicated and expensive transmission gear in order to drive machinery at a suitable speed.

When the tail-race level is subject to variations, the wheel must be installed so as to just clear the high water level in it, for as is easily seen, the wheel turns in the opposite sense to the flow in this canal, and if the wheel dipped into the water, the buckets could not empty themselves and the water would be raised behind, thus causing a loss of energy with the risk of damaging the wheel.

This can be remedied by turning back the intake canal so as to reverse the wheel, or by changing the direction of the tail-race. Then the wheel may be immersed to a depth of 4 or 5 inches during times of high water, since it turns in the direction of the current of the tail-race.

To sum up, this wheel has a high efficiency, sometimes even greater than 80 per cent.; but it is the slowest and largest wheel for a given fall.

30. Overshot bucket wheel with head of water. This wheel is constructed exactly as the last, and works in the same manner, that is to say, in virtue of both the kinetic energy and weight of the water.

It differs essentially however in the manner in which the water is supplied. The canal, instead of being open at the end, is provided with a sluice gate B which can be raised or lowered so as to control the quantity of water flowing, or stop it altogether (fig. 20).

While working, the gate is raised to a certain height E; the depth of water in the canal above the sill of the sluice is called the *head of water*.

The canal is prolonged by a channel possessing a slight slope to compensate for frictional resistance, in this channel the water flows in the form of a sheet of uniform thickness e, but reduced by the contraction in passing the sluice gate so that $e = 0{\cdot}8\mathrm{E}$.

The velocity of the water is given by the expression:

$$\mathrm{V}_0 = \sqrt{2g(\mathrm{Z} - e)} = \sqrt{2g(\mathrm{Z} - 0{\cdot}8\mathrm{E})}.$$

In practice the vertical opening E of the gate should be from 2 to 6 inches.

The efficiency of this wheel is obtained by the same considerations as for the last; it is always necessary to add the loss of fall $\mathrm{P} \times i \times l$, resulting from the difference in level between the two ends of the channel, namely the slope i multiplied by the length l of this channel.

The depth Z to be given to the head of water depends on the flow of the stream and the variations in the up-stream level. For falls of from 10 to 25 feet, Z should vary from 2 to 3 feet about.

If the level of the upper reach and the flow are constant, the volume flowing per second will be given by the expression:

$$q = m\mathrm{E} \times \sqrt{2g(\mathrm{Z} - 0{\cdot}8\mathrm{E})}.$$

This formula gives values of q varying from 1·27 cu. ft. to

1·58 cu. ft. for E = 2 inches, and from 3·53 to 4·5 cu. ft. for E = 6 inches as Z varies from 2 to 3 feet.

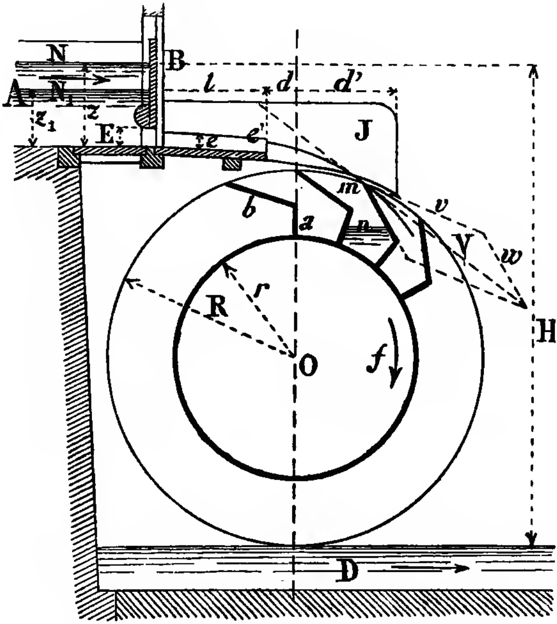

Fig. 20. Overshot wheel with head of water.

The width l of the wheel will be obtained from:

$$l = \frac{Q}{q},$$

where Q is the total flow into the machine.

The value of m may be taken equal to 0·7 at least if the gate has a curved foot as shown on the drawing, in order to reduce the contraction.

If the level of the millhead vary from N_1 to N and the volume of flow vary from Q to Q′ between low water and high water, the condition might be imposed at the outset, that the values of the speed V_0 corresponding to these two

extreme states, should be limited to a ratio K which ought not to exceed 2; then we have:

$$\frac{\sqrt{Z - 0\cdot8E}}{\sqrt{Z_1 - 0\cdot8E_1}} = K.$$

If further we put:

$$\frac{Q'}{Q} = \frac{q'}{q} = K',$$

then: $$q = mE_1 \times \sqrt{2g(Z_1 - 0\cdot8E_1)},$$

and $$K' \times q = q' = mE \times \sqrt{2g(Z - 0\cdot8E)}.$$

From which dividing the two equations member for member, is deduced:

$$K' = K \times \frac{E}{E_1},$$

and from this, K and E being given, E_1 is obtained.

As for the value of Z it will be given by the equation:

$$Z = \frac{K^2 \times d + 0\cdot8E\left(\frac{K^3}{K'} - 1\right)}{K^2 - 1},$$

where d is used to denote the known difference of levels N and N_1.

Knowing Z, Z_1 can be deduced with the aid of the foregoing equation, as also can q, q' and l.

The diameter 2R of the wheel is obtained from the formula:

$$2R = H - (Z + il + \tfrac{1}{12}),$$

in which $\frac{1}{12}$, or 1 inch is the vertical distance between the end of the channel and the summit of the wheel.

The absolute velocity V of the water entering the buckets varies from 10·1 to 13·5 feet per second for $Z = 2$ feet and $E = \frac{1}{2}$ foot on the one hand, and $Z = 3$ and $E = \frac{1}{6}$ foot on the other hand.

The peripheral speed v of the wheel cannot be greater as a

maximum than half the absolute velocity of the entering water, namely:

$$v = \frac{V}{2},$$

this speed varies then in the stated conditions from 5·0 to 6·7 feet per second as maxima.

The number of turns n of the wheel per minute is expressed by:

$$n = \frac{v \times 60}{2\pi R}.$$

The inside width l' of the wheel and the depth $(R - r)$ of the buckets is calculated as in the last case.

This type of wheel is used for falls varying from 10 to 40 feet. Nevertheless it is not suitable for falls of less than 13 feet. In this case the angular velocity being relatively great with this wheel, the free surface of the water in the buckets takes a very pronounced curve in virtue of the centrifugal force, which prevents the buckets from holding much water.

This wheel is particularly suitable for use when the level of the supply or the flow of the stream is variable. It is very suitable for large flows, for it can deal with about twice the quantity of water that can be dealt with by the wheel without the head of water.

Finally although its efficiency is less, its greater speed makes it preferable in the greater number of cases. In fact it is not nearly so bulky, it requires less complicated transmission mechanism, it has more regulating effect on the speed under varying load, and its erection is very much more economical on account of the reasons stated.

When the tail-race level is variable, the wheel may always be drowned to the extent of 4 or 5 inches by adopting a reversed feed arrangement at the end of the supply channel which directs the sheet of water backwards reversing the direction of rotation of the wheel. The supporting wall may then be so shaped as to form a circular channel reaching to

the base of the wheel and retaining the water in the buckets until this lowest point is reached.

31. Breast wheel with shuttle gate. This wheel is characterised by the manner in which the water is supplied to the buckets. The water instead of entering the wheel at the top, as in the preceding arrangements, enters at the side a little above the centre (fig. 21).

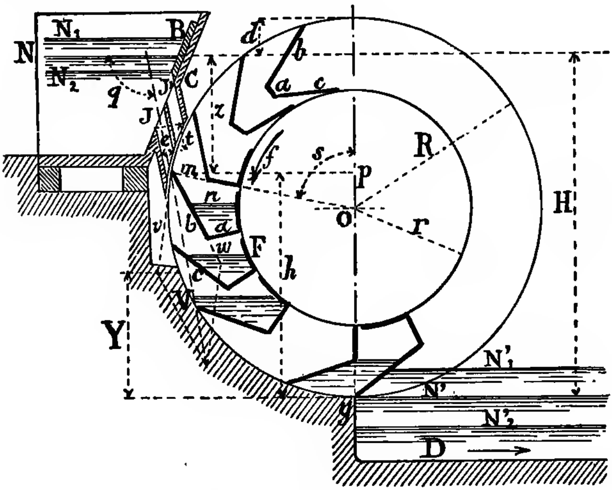

Fig. 21. Breast wheel.

As in the case of the overshot wheel with the head of water, the intake canal ends in a trough closed by an inclined plane BC which is pierced with a number of holes provided with extensions or guide-plates J, J′, the amount of opening being regulated by the gate B.

These extensions cause the water to enter the buckets in the most favourable direction for good efficiency.

The form of the buckets may be sensibly the same as in those wheels already described, but it is necessary to have vents in the sole-plate F. These vents are necessary because

of the manner in which the water enters the wheel, for the mouth of the bucket becoming covered by the guide-plates, the air must be provided with an outlet through the sole-plate.

The sole-plate may be done away with entirely, and the buckets be formed of polygonal pockets such as b, a, c, which would fit one inside another, the vent holes being naturally formed by the spaces left between the inner sides of consecutive buckets, and these holes thus being at the top of the buckets, the latter may be completely filled with water.

This wheel naturally requires the use of a breast which is built in the base of the retaining wall. In fact, the use of the breast is most important with this wheel in maintaining the water in the buckets until the lowest point is reached, since even with its aid the water only acts on the wheel during a quarter of a revolution.

The wheel turns in the same direction as the current in the tail-race and therefore no inconvenience results from drowning the lower buckets to the extent of 4 or 6 inches.

The various losses of power are due to the following causes: (1) To friction in the guide-plates; this loss is kept low by giving the guides only just the necessary length to shape the water jets to the sections J, J′;

(2) To the relative velocity w with which the water enters the bucket, the corresponding kinetic energy being a pure loss in eddies within the bucket;

(3) To the loss of fall as measured by the vertical distance between the point m where the water enters the bucket and the level n which it takes when inside;

(4) To the clearance between the wheel and the breast which allows water to escape after it comes up to the edge c of the bucket. This water does no work as its weight does not act on the buckets;

(5) To the friction of the water flowing over the surface of the channel with the peripheral speed v of the wheel;

(6) To the loss of fall resulting from the water in the lower bucket falling to the tail-race level N′ without doing any useful work;

(7) Finally, to the speed with which the water is discharged, which is the circumferential speed of the wheel.

The energy losses due to the relative speed and to the peripheral speed are expressed respectively by $\frac{Pw^2}{2g}$ and $\frac{Pv^2}{2g}$. Now it was seen in the consideration of the wheel without the head of water, that these losses are a minimum when the peripheral speed v is half the projection of the absolute speed V on the direction of the peripheral speed itself.

These considerations enable us as before to determine the most rational form to give to the buckets. For the velocity V being given in magnitude and direction, it suffices to project this speed on the tangent at m and to take half this projection for v; in this way the parallelogram of velocities is constructed, and the relative velocity w thus obtained also gives the direction of the piece b of the bucket.

Under these conditions, since the relative velocity of the water on entering the bucket is made parallel to the outer side of the bucket, the water enters without shock.

The two principal losses of work, due to the fall lost both on entering the bucket and on leaving the lower bucket, depend on the coefficient of filling and consequently on the depth of the buckets which is practically equal to $(R-r)$, the difference between the outer and inner radii of the wheel. Therefore $(R-r)$ should be made as small as possible consistent with satisfying the conditions already specified.

As for the loss caused by the water which leaks between the wheel and the channel, it is evident that it increases with the amount of clearance and with the speed of the wheel; for this reason too great a speed of rotation should not be adopted and the clearance should be reduced to a minimum. Practically this clearance cannot be less than $\frac{3}{16}$ of an inch.

Finally, the loss resulting from the friction of the water held back by the breast depends on the length of the breast and the square of the speed of the wheel; consequently it does not pay to adopt too great a filling coefficient or too great a speed for the wheel.

Taking into account these various sources of loss even when their combined effect is a minimum, it is found that the efficiency of the breast wheel lies between 65 and 70 per cent., according to the total height of fall H, the value adopted for the absolute speed V and the angle which this speed makes with the circumferential speed of the wheel.

This type of water-wheel is suitable for falls of from 8·5 to 13 feet; it is particularly suited to low falls where overshot wheels cannot be employed and when the head-race level, flow and tail-race level are variable, as with sluice gate regulation.

The number of openings is varied according to the height of the head-race level and the volume to be delivered.

If q be used to denote the volume per second per foot of width corresponding to high water and head-race level N_1, Z_1 being the depth of the lower end of the nozzle below the level N_1, and E_1 the thickness of the sheet of water pouring into the wheel, the following relation holds:

$$q = mE_1 \times \sqrt{2gZ_1},$$

where m is the coefficient of contraction which may vary from 0·75 to 0·85 according to the shape of the guide-plates.

Similarly, for low water with the head-race level at N_2, putting as before $q' = K \times q$ for the corresponding volume of flow, we have:

$$K \times q = mE_2 \sqrt{2gZ_2}.$$

E_1 may be fixed arbitrarily and then E_2 deduced from it, provided that the two values lie between $2\frac{1}{2}$ and 8 inches.

A certain connection must hold between E_1 and E_2 such that the extreme velocities of flow V do not differ too greatly; this connection may be expressed as a function of K as follows:

$$\frac{K}{K'} = \frac{E_1}{E_2},$$

where K′ is arbitrary, but should not exceed the value 2.

Having fixed E_1 and E_2 according to these various considerations, one might calculate the values of Z_1 and Z_2 if the

discharge q is given, by making use of the above expressions. Generally, the head of water measured to the mean level of the head-race N is taken as 1·6 feet which corresponds to $V = 10$ feet.

To determine the radius R of the wheel, the following empirical rule may be used:

$$2R = H + 40 \text{ inches.}$$

The angle which the outer portion of the bucket makes with the tangent to the circumference is generally comprised between 25 and 30 degrees.

The absolute velocity at the outlet of the guides is given by:

$$V = \sqrt{2gZ}.$$

In order to find its direction it is sufficient to make the inclination (namely the tangent of the angle) between V and the tangent giving the direction of v, half the inclination which this tangent makes with the piece bm of the bucket. This construction corresponds to the condition for maximum efficiency of the wheel.

Then the magnitude of v is obtained as we have seen above by taking half the projection of V on the direction of v. The number of turns per minute is easily deduced from the expression:

$$n = \frac{v \times 60}{2\pi R}.$$

The breadth l and the depth $(R - r)$ of the buckets are obtained by the same method as explained in the former example.

M. Millot has so modified the breast wheel that the breast is not needed (fig. 22). To this end the supply canal divides into two branches which curve round to the inner side of the wheel so that the water enters the inner circumference. But this wheel is difficult to construct and can only be used for small powers because the arms must be placed in the middle section of the wheel, instead of being fixed to the flanges, in

consequence of the arrangement of the feed troughs, and therefore the breadth is very limited, being 5 feet at most.

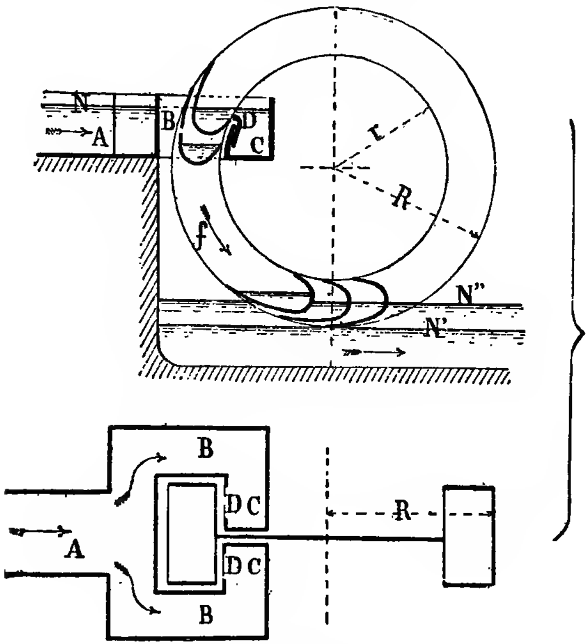

Fig. 22. Millot's wheel.

32. Breast Wheel without head of water. In this class of hydraulic motor, the water is fed in at the side below the level of the centre of the wheel (fig. 23).

For this purpose the tail-race is prolonged by a circular bed *k* and two cheeks *h*, which are continuations of the sides of the said tail-race. The wheel buckets, composed of the blades proper *c* and their supports *s*, are not provided with flanges so that the water which enters the wheel is kept there by the circular bed and the cheeks.

The circular channel is continued on the up-stream side by a cast-iron plate, or *swan's neck*, which is placed in a

trench and is built into the walls of the canal. The moveable gate *a* slides over this fixed piece and may be raised or

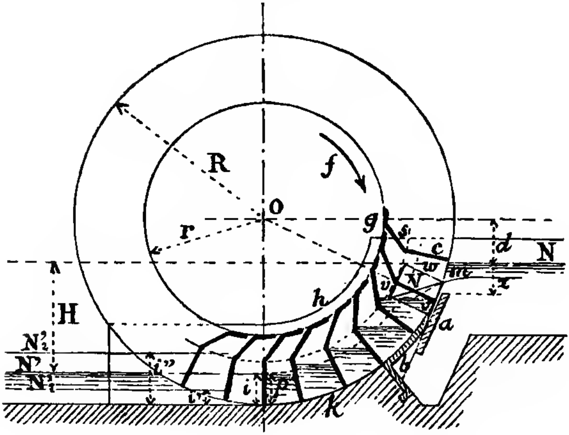

Fig. 23. Side wheel without head of water.

lowered according to the variations in the head-race level. This gate acts as a dam, the feed water flowing over it in the form of a sheet of varying thickness *z*.

The height of the centre O of the wheel above the penstock level varies from 20 inches as a minimum for the highest falls to 40 inches as the maximum for the lowest falls.

Obviously the water enclosed in the buckets finds various levels so that the buckets themselves constitute communicating vessels, the communication taking place by means of the clearance existing over all the outer surface of the wheel corresponding to the depth *p* of water in the buckets.

It should also be observed that a certain number of buckets on the left of the vertical diameter of the wheel are drowned in the tail-race. Now the tail-race surface may be at the same height as that of the water in the lowest bucket *p*, at N′; or at a lower level N_1'; or lastly at a higher one N_2'. Evidently the latter two conditions are unfavourable to high

efficiency, because in the first case water falls in the lowest bucket from level p to level i' without doing work on the machine, and in the second case the tail-race water exerts an opposing force reducing the motive force on the wheel.

Again summing up the various causes of loss of energy produced in the working of this wheel:

(*a*) The first loss is due to the friction of the water passing over the gate; this loss, however, which depends on the square of the relative feeble velocity of the flowing water is quite negligible.

(*b*) The second loss is due to the relative velocity w with which the water enters the buckets; this velocity is wasted in eddies and only a very small part of the corresponding kinetic energy is given to the inclined portion of the blade s on to which the water rises, spending a fraction of its kinetic energy in so doing; however, practically the whole energy $\frac{Pw^2}{2g}$ corresponding to this speed is lost.

(*c*) The third loss occurs through water flowing from one bucket into the next below it between the blades and the channel and cheeks, as the result of difference of level in these communicating vessels.

(*d*) The fourth loss results from friction between water and the wheel channel; this loss is proportional to the square of the circumferential velocity of the wheel and also to the developed area of the channel and wheel, and consequently the radius.

(*e*) The fifth loss results from water leaving the buckets with the speed v of the wheel's circumference; the energy wasted in this way is equal to $\frac{Pv^2}{2g}$.

(*f*) Finally there is a last loss resulting from either insufficient or excessive immersion, as has been seen already.

If the level i' is lower than p the loss due to the difference of level is:

$$P(p - i').$$

If, however, i'' is higher than p the wheel exerts force on the water of the tail-race which, reacting on the wheel, opposes its motion; the resulting loss of work depends on the amount of excessive immersion and on the speed of the wheel.

These last losses are caused whenever the depth i of the water in the channel is either greater or less than the depth p of water in the lowest buckets, on account of variations in tail-race level.

The sum of the losses due to causes b and e equals $\frac{P}{2g}(w^2+v^2)$; the minimum value of this expression differs according to whether the circumferential speed of the wheel is fixed, or whether V is given, which comes to the same thing as being given the thickness of the sheet of water passing over the tip of the gate.

In the first case the loss is least when the velocity V is equal to the projection of v on its direction; here the relative velocity w is perpendicular to V, and the blade which should be parallel to w, approaches the vertical, which is a form very unfavourable to its emergence.

When V is given, the condition for minimum loss is that the speed v of the wheel's circumference should be equal to half the projection of V on its direction, which enables the blade to be made parallel to the direction of the third side w of the triangle.

Actually the blades are neither vertical nor as oblique as would result from this latter construction; generally the outer portion is in line with the radius of the wheel, that is in a direction intermediate to the last two; hence the three speeds form a triangle and the speed v becomes equal to the total projection of V on its direction; from this, as a consequence of the geometrical relation between the three sides of a right-angled triangle, the loss under consideration becomes:

$$\frac{P}{2g}(w^2+v^2)=\frac{PV^2}{2g}.$$

The loss (c) is made a minimum by reducing the clearance

between the wheel and its channel as much as possible; but this can scarcely be less than $\frac{3}{16}$ of an inch from a mechanical point of view if all risk of the wheel rubbing against the sides or the bottom of its channel is to be avoided.

We saw that the loss (d) diminished with the radius of the wheel and with the speed v. Obviously, since the lost work is absorbed in friction, this loss may be made a minimum by making the channel as regular and smooth as possible.

The absolute velocity V with which the water enters the buckets depends on the depth z of the sheet of water. If the dam be sharp-edged as shown in figure 23, the formula giving the speed will be:

$$V = \sqrt{2g \times 0{\cdot}57z},$$

the coefficient 0·57 being found by experiment.

If on the other hand the gate form a dam with thick walls as in the case of the overshot wheel without the head of water, the following relation holds:

$$V = \sqrt{2g\frac{z}{3}}.$$

In practice the values of z ought to lie between the limits 7 and 12 inches.

The volume of water flowing per foot of width of the dam per second is given by the relation:

$$q = mz \times \sqrt{2gz},$$

in which m is a contraction coefficient which for the gate shown in the above-mentioned figure is 0·41; the coefficient becomes equal to 0·45 when the gate is provided with a curved edge, and may be as low as 0·385 when the thickness of the dam is at least equal to 1·5z.

As the values of z vary from 7 to 12 inches, the values of q vary themselves from 1·37 to 3·6 cubic feet according to the nature of the dam.

The radius of the wheel from the accompanying figure is:

$$R = d + H + i.$$

d and i are first fixed approximately, and then, knowing z, the position of the mean stream line is deduced, this is $0{\cdot}57z$ below the penstock level for a sharp-edged dam, and $\frac{2}{3}z$ for one with thick edges.

Then the parabolic track of the mean stream-line is traced, and the tangent to the parabola at the point where this parabola cuts the exterior circumference gives the absolute velocity V of the water on entering the buckets. The velocity v of the wheel is deduced by the triangle of velocities in the manner we have already seen.

The depth $(R - r)$ of the buckets is obtained from the equation:

$$Q = 0{\cdot}90 \text{ to } 0{\cdot}95\, K \times l\,(R^2 - r^2) \times \frac{v}{2R},$$

in which Q is the total volume of flow per second and K is a coefficient representing the fractional filling of the buckets, which varies from $\frac{1}{2}$ to $\frac{2}{3}$.

$(R - r)$ having been obtained from this relation, it may then be seen whether p is equal to i, since it is known that:

$$p = 0{\cdot}90 \text{ to } 0{\cdot}95\, K \times (R - r).$$

If p is not the same as i, the calculation should be recommenced, using a new value for i, until at last the condition $i = p$ is satisfied. This condition, however, is no longer satisfied when the tail-race level varies, and a loss of work will necessarily result.

Side wheels are suitable for falls of from 3 to 9 feet. They are satisfactory when the head-race level varies but little: the inconvenience arising from an appreciable rise of level may be obviated, however, by the use of a circular gate whose upper edge is so arranged as to be always as close as practicable to the outer circumference of the wheel.

The peripheral speed of the side wheel varies from 3 to 5 feet; this relatively slight speed renders it unsuitable except for machinery whose resistance to motion does not change much, such as milling, spinning, and pumping machinery.

The efficiency of the side wheel varies from 60 to 75 per cent. as the fall increases from 3 to 9 feet.

33. Side Wheel with head of water. This wheel is constructed like the side wheel without the head of water; but the penstock has at its end a gate *b*, under which the water passes into the wheel (fig. 24).

This gate is kept at such a distance *e* from the bed of the channel that the section of the sheet of water feeding the wheel would be equal to $l \times e$ if there were no coefficient of contraction to be taken into account in the passage under the gate. It is advisable to incline this gate until at its lower edge it is tangential to the wheel in order to reduce the contraction coefficient as much as possible.

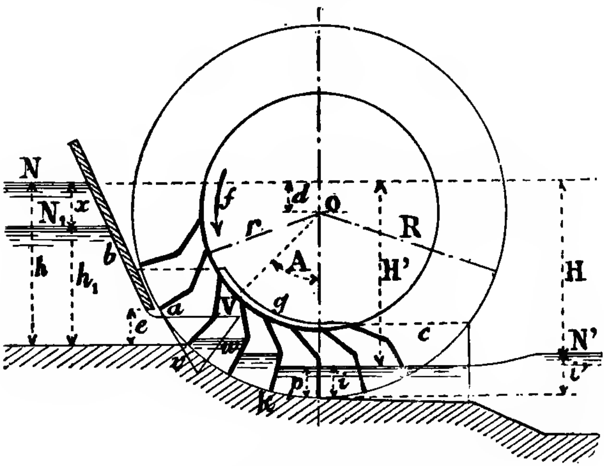

Fig. 24. Side wheel with head of water.

The bed of the penstock should be slightly inclined towards the wheel, for the water to keep the velocity it gains in passing under the gate.

The total depth of the water in the penstock constitutes the head of water.

It will be observed that the bed of the circular channel is continued by a straight part sloping gently towards the tail-race and at the end of the side-cheeks C the bed suddenly dips to the bottom of the river. This contrivance causes a standing wave and allows the water to rise after leaving the wheel while keeping the surface of the water between the cheek-boards at a lower level.

The losses of energy occurring in the working of this wheel are analogous to those which have been considered in the case of the wheel without the head of water.

The first loss, due to friction in the passage of the water under the gate and over the sill of the penstock, is a very slight fraction of the energy corresponding to the speed V of the mean stream-line whose velocity head (the head corresponding to and producing the velocity) is $(h - 0{\cdot}8e)$.

The second, due to the relative speed w with which the water enters the wheel, is always equal to $\frac{Pw^2}{2g}$.

The third part relates to the flow of water through the clearance existing between the wheel and its channel.

The fourth is due to friction against the channel.

Lastly, the water leaves the buckets possessing the velocity v of the wheel, which causes a loss equal to $\frac{Pv^2}{2g}$.

There may be in addition, either an insufficient or an excessive immersion causing a further loss of energy as in the preceding case.

In that case it was seen that on fixing the circumferential speed of the wheel and seeking to make the sum of the losses expressed by $\frac{P}{2g}(w^2 + v^2)$ a minimum, a direction was given to the blade which was very unfavourable to its emergence. In the wheel now under consideration this condition would conduce to a direction of the blades still more impracticable.

Actually then, the velocity V of the water entering the wheel should be fixed; this is the same as the speed under

the gate, and comes to the same thing as fixing the depth of the head of water which generates the speed V.

Here it is always advisable to arrange the paddle parallel to the direction of the relative velocity w. In fact if the outer portion of the paddle is made to be in line with the radius as is done by many makers, the loss of energy under consideration becomes equal, as has been seen, to $\frac{PV^2}{2g}$, that is to say to a loss of fall equal to the whole head generating the absolute velocity of the water.

Now for the wheel without the head of water this velocity head is only a comparatively slight fraction of the total fall, while for the wheel with head of water it may attain to $\frac{2}{3}$ of this fall, which consequently corresponds to a very serious loss of energy.

The efficiency of the wheel with which we are now dealing, varies from 40 to 55 per cent. according to the available fall.

The depth h of the head of water will be greater in proportion as the fall is higher, but it increases at a slower rate than the fall. Thus for the value of H equal to 3 feet, h will be equal to $\frac{2}{3}$ H, that is 2 feet; for H = 6 feet, the depth h will be very nearly $2\frac{1}{2}$ feet or about $\frac{5}{12}$ of H.

When the penstock level varies, the opening under the gate must necessarily be altered; let e be this opening for the highest level N and e_1 be that for the lowest level N_1.

Using the same notation as in former analogous cases, we have:

$$\frac{Q_1}{Q} = \frac{q_1}{q} = K,$$

$$Kq = q_1 = me_1 \sqrt{2g\,(h_1 - 0{\cdot}8e_1)},$$

$$q = me \sqrt{2g\,(h - 0{\cdot}8e)};$$

and:

$$\frac{q_1}{q} = \frac{e_1}{e} \times \frac{\sqrt{h_1 - 0{\cdot}8e_1}}{\sqrt{h - 0{\cdot}8e}}.$$

Putting for the sake of simplicity:

$$\frac{\sqrt{h_1 - 0{\cdot}8e_1}}{\sqrt{h - 0{\cdot}8e}} = K',$$

then:
$$K = \frac{e_1}{e} \times K' \quad \text{or} \quad \frac{e_1}{e} = \frac{K}{K'}.$$

Fixing K′, which should lie between 0·6 and 0·9, the fraction $\frac{e_1}{e}$ may be deduced from this, which enables either e_1 or e to be calculated according to which of these two is given. In practice the gate should always be raised between 8 inches as a minimum and 16 inches as a maximum.

After this calculation has been completed, h is obtained from the equation:

$$h = \frac{x + 0{\cdot}8e \times \left(\frac{K}{K'} - K'^2\right)}{1 - K'^2},$$

x being the known difference of level between the two extremes N and N_1.

From this h_1 and also q and q_1 are deduced. As for the coefficient of contraction m, this depends on the inclination of the gate and varies from 0·74 to 0·80 as this inclination diminishes from 63 to 45 degrees.

For the extreme values of h and e, the volume of flow per second per foot of width of the wheel with the head of water is as a minimum:

$$q = 0{\cdot}74 \times \tfrac{2}{3} \times \sqrt{2g\,(2 - 0{\cdot}8 \times \tfrac{2}{3})} = 4{\cdot}8 \text{ cu. ft. per sec.},$$

and as a maximum:

$$q' = 0{\cdot}8 \times \tfrac{4}{3} \times \sqrt{2g\,(3 - 0{\cdot}8 \times \tfrac{4}{3})} = 11{\cdot}8 \text{ cu. ft. per sec.}$$

As the penstock level varies, the speed V of the water entering the buckets increases or diminishes, and with it the speed of the wheel itself. If the machinery driven by the hydraulic motor requires a constant speed, V must be maintained constant. To do this, the single gate is replaced by two vanes each provided with an independent raising mechan-

ism ; these two vanes are then manipulated in such a manner that the centre of the orifice is maintained at a constant depth below the varying penstock level so that the generated velocity is constant.

As regards the diameter to be adopted for the wheel, a large diameter is favourable to high efficiency; in practice this diameter lies between 12 and 24 feet.

The circumferential speed v of the wheel varies from 6 to 10 feet.

The depth of the buckets may increase from 16 to 40 inches when the flow is very variable.

The position of the wheel centre is so fixed that the condition $i = p$ is realised for the average state of affairs.

The speed of the wheel with the head of water being at least double that of the one without the head of water, the loss due to the speed with which the water leaves the wheel being proportional to the square of this velocity, is four times as great as in the former case. It is because of this that it is advisable to use this speed to raise the water to the height of the tail-race surface, by forming a standing wave.

As a consequence the wheel works as if the fall were actually

$$H + i' - i = h,$$

that is to say with a greater fall than H.

In this way half the height $\frac{v^2}{2g}$ corresponding to the speed v may be gained, and the saving in efficiency may be 10 per cent. or even more from this cause.

The side wheel with the head of water is employed where the head-race level and the flow are very variable ; when the volume to be dealt with per second is very great, since for equal flows the wheel with the head of water is smaller in size on account of its higher speed ; when the power of the fall is superabundant, for the same reasons, and further because in this case part of the efficiency may be sacrificed ; lastly when the driving force required for the machinery varies, the high speed of the wheel enabling it to act as an efficient flywheel.

34. The Sagebien Side Wheel. The buckets of this wheel are formed by flat vanes *b* whose planes are not in line with the radii, but are tangential to a horizontal cylinder concentric with the axis of the wheel (fig. 25). The depth of the bucket ring is relatively large, there is no sole-plate, the buckets forming a sort of vessel open both top and bottom, and into which the water penetrates as they dip into it.

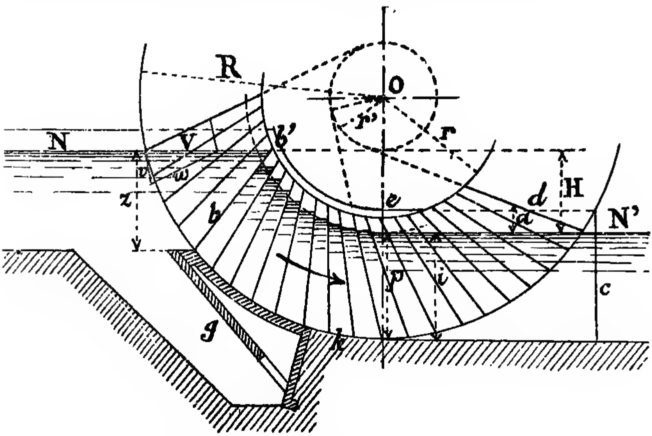

Fig. 25. Sagebien Side Wheel.

The wheel turns in a circular channel *k* prolonged by a swan's neck against which the submerged vane *g* slides. Generally this vane in its lowest position has its upper edge level with the bed of the penstock.

The side cheeks of the channel are continued down stream to the point where the wheel leaves the tail-race.

Obviously the buckets as they first dip into the supply channel behave as communicating vessels in which the water takes the same level as the surface N of the penstock. In fact the water enters the bucket ring with a speed sensibly the same as that of the current in the penstock, and since this speed is always very low, the blades plunge into the water without encountering any appreciable resistance.

As the paddle comes toward b, the swan's neck begins to obstruct the lower opening until at last the communication with the penstock is interrupted.

Apart from the leakage occasioned by the clearance between the wheel and its channel, the vessel formed by the paddles keeps its charge until a point vertically below O is reached, where the water in the bucket attains the same level N′ as the tail-race surface.

The water exerts no further motive force on the buckets after this point is passed; with the wheel still turning, the vessel emerges little by little from the tail-race and is completely empty by the time d is reached.

The wheel therefore in carrying the water from the head-race to the tail-race acts as a water meter and its speed is proportional to the flow of water.

It may be observed that the inclined position of the blades is not very favourable to their emergence on the down-stream side; but as the speed of the wheel scarcely ever exceeds 2 feet or 2 feet 8 inches per second at the circumference, there is no appreciable resistance to the emergence on the part of the water.

The energy losses produced in this wheel are again analogous to those studied in the previous cases.

The loss which depends on the quantity $(W^2 + v^2)$ becomes reduced to the value $\frac{Pv^2}{2g}$ when the most favourable conditions for high efficiency are observed, the speed v being given.

Now since v is never greater than 2·7 feet per second, in taking this extreme speed the corresponding loss of fall is only equal to:

$$\frac{v^2}{2g} = \frac{7{\cdot}3}{2 \times 32} = 0{\cdot}114 \text{ ft.}$$

So that for a fall $H = 3$ feet this loss at its greatest would be only about 3 per cent.

As the losses of energy resulting from clearance and from friction in the passing of the water over the channel, are respectively proportional to the speed v and to the square of

the speed v^2, obviously these losses are reduced as much as possible in the Sagebien wheel whose speed is always relatively small.

It follows that the efficiency of this wheel is very high in practice and varies from 80 to 90 per cent. according to the height of the fall and the dimensions of the wheel.

The mean speed u of the penstock current being between $1\frac{1}{3}$ and 2 feet per second, and being known in every case, the thickness Z of the sheet of water is easily determined. The penstock is rectangular in section, its breadth l being determined from the relation $l = 2 \times Z$, which corresponds as we know to the minimum loss per foot in friction against the sides and bed.

If Q be the volume per second, the section of the stream over the vane being $l \times Z$, we have:

$$Q = l \times Z \times u,$$

and replacing by l its value 2Z:

$$Q = 2Z^2 \times u,$$

whence:

$$Z = \sqrt{\frac{Q}{2u}},$$

Z may reach or surpass 5 feet without inconvenience.

When the penstock level and the flow are variable the vane should be raised or lowered to suit.

The radius R of the wheel is comprised between 10 and 20 feet, according to the height of the fall, the magnitude of the flow, and the variableness of this flow and of the tail-race level. In all cases the angle α which the plane of the paddle d makes with the surface N′ of the tail-race ought not to be less than 30 degrees.

The speed v being chosen, and the speed V with which the water enters the wheel horizontally being very nearly equal to u, the velocity in the penstock, the triangle of velocities is drawn for the point at which the penstock level N cuts the outer circumference of the wheel. In this way the relative speed w is obtained which gives the direction of the paddle

and consequently determines the radius of the cylinder to which the different blades are to be made tangential.

The number of turns per minute is, according to the general formula:

$$n = \frac{60v}{2\pi r}.$$

The number of turns being proportional to the circumferential speed of the wheel which is very small, and inversely proportional to the radius which on the other hand is relatively large, it is easily seen that the Sagebien wheel has only a slight angular velocity. The number of turns scarcely ever exceeds 2·5 and may be below 1 turn per minute.

The depth $(R - r)$ of the buckets, taking into account the space occupied by the thickness of the blades, is calculated from the formula:

$$Q = 0{\cdot}90 \text{ to } 0{\cdot}95\,(R^2 - r^2) \times \frac{v}{2R}.$$

It is necessary in every case that the value found for $(R - r)$ and consequently for R should be such that the blade b whose lower extremity is level with the tip of the vane, will have its extremity b' situated 2 or 4 inches above the penstock high water mark, so that the water cannot well over into the inner portion of the wheel.

In this wheel, as in other wheels which run partially drowned in the tail-race, losses of energy are occasioned through either insufficient or excessive immersion, when in consequence of variations of the tail-race level and the flow, the depth p of the water in the lowest bucket is greater or smaller than the depth i of the water in the tail-race.

The number of blades is got from the relation:

$$N = \frac{2\pi R}{E}.$$

The spacing E of the blades, measured on the outside of the wheel, should be comprised between 14 and 16 inches.

It is convenient moreover for the number of blades to be an exact multiple of the number of arms in the wheel.

The Sagebien wheel is used for small falls of from 2 to 9 feet and is suitable for large flows.

In fact the volume per second per foot of breadth varies from 8 to 14 cubic feet, according to the speed, the depth of the paddles, and the diameter.

It is characterised by its small angular velocity which renders it unsuitable except for installations where the machinery runs slowly and opposes uniform resistance to driving.

The flow of this wheel is proportional to its speed, and this fact specially adapts it for raising water, the work of the pumps being proportional to the speed, and consequently to the motive power of the hydraulic wheel.

Conversely it is ill adapted for driving a machine with varying resistance such as a rolling-mill, because during the rolling the machine slows up, and the power of the wheel diminishes just when increased effort is necessary, and *vice versa*.

Moreover the very low speed of this wheel makes expensive transmission gear necessary in most cases, this gearing absorbs part of the energy given out by the motor and to some extent neutralises its high efficiency. Finally its large dimensions and deep buckets make its construction and installation very troublesome.

35. Undershot Wheel with Flat Blades. In this wheel the blades point towards the centre, there is no second portion to the blade, and the wheel has neither any inner rim nor flanges, but turns between two side cheeks *j*. Further the wheel is set in a circular channel S, over an arc of trifle greater radius than the circumference (fig. 26).

The gate in which the penstock ends is inclined at about 45 degrees to the horizon, so that its lower edge is as close as possible to the wheel rim.

Water flows under the gate through the opening *e*, then glides over a slightly inclined bed, in order to enter the wheel without loss of velocity. If it were not for the slight inclina-

tion given to this part of the bed, the velocity v due to the head of water h would be reduced in consequence of friction.

The water rises in the bucket to a certain height p which is practically equal to the immersion i in the tail-race.

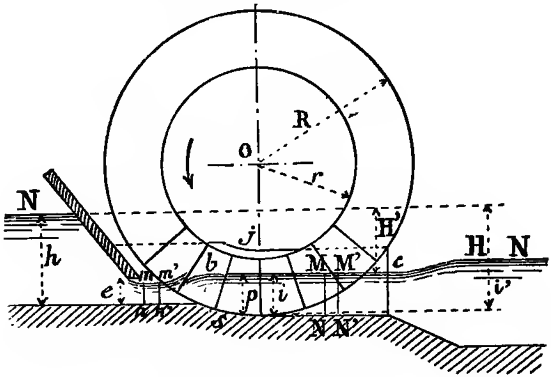

Fig. 26. Undershot wheel.

As the result of setting the wheel in the circular portion of the channel, all direct communication between the head-race and the tail-race is avoided during the working course of the buckets, that is until the blade of the bucket under consideration has come into the vertical position.

The circular channel is continued by a gently inclined bed so that the water maintains its velocity after leaving the wheel. Just beyond the end of the side cheeks, the canal bed dips suddenly so as to cause a standing wave which raises the water level by an amount $(i' - i)$ to the level N′ of the tail-race.

To estimate the work received by the wheel, it is to be observed that the quantity P of water which passes through the motor per second, has a velocity V on passing the section mn, and possesses only the reduced speed v in the section MN situated farther on.

Consequently the change of momentum in the mass of

water $\frac{P}{g}$ between the two sections considered, during one second, is:

$$\frac{P}{g} \times (V - v).$$

Now this change is equal to the sum of the horizontal forces acting on the aforesaid mass of water.

The force of gravity being vertical has no effect on the horizontal motion of the liquid mass; therefore only the difference in the hydrostatic pressures at the sections *mn* and MN on the one hand, and the resultant F of the wheel-blade reactions on the column of water on the other hand, have to be taken into consideration.

Using S and S′ to denote the area of the transverse sections of the column at *mn* and MN respectively, and y and y' to denote the depths at these same sections, also letting d represent the density of the liquid, the difference in the hydrostatic pressures is given by the expression:

$$\frac{d}{2}(yS - y'S').$$

And writing the known law connecting change of momentum with the impulse of acting forces during unit time:

$$\frac{P}{g}(V - v) = F - \frac{d}{2}(yS - y'S'),$$

in which the sign $(-)$ results because the bracket term $(yS - y'S')$ is negative since $y'S'$ is necessarily greater than yS.

This expression may be written:

$$F = \frac{P}{g} \times (V - v) + \frac{d}{2} \times (yS - y'S').$$

Now the force F is precisely the resultant of the horizontal forces exerted on the wheel; therefore the power received by the wheel is obtained by multiplying this force by the speed v and:

$$Fv = \frac{Pv}{g} \times (V - v) + \frac{dv}{2}(yS - y'S').$$

And observing that the weight of water passing each section per second is constant, namely:

$$P = dSV = dS'v \quad \text{and} \quad yV = y'v,$$

it is found on making use of these relations that the above formula becomes:

$$Fv = Tu = \frac{Pv}{g}(V - v) + \frac{Py}{2}\left(\frac{v}{V} - \frac{V}{v}\right).$$

As V is greater than v, it is seen that the quantity $\left(\frac{v}{V} - \frac{V}{v}\right)$ is always negative and that the second term of the second member is consequently subtracted.

Just as in every analogous case with a gate and a head of water, we have:

$$y = 0{\cdot}8e,$$

and:

$$V = \sqrt{2g\,(h - 0{\cdot}8e)}.$$

It has been demonstrated experimentally that the maximum useful work Tu is done when the ratio of the wheel's speed to the speed at which the water enters the wheel, $\frac{v}{V} = 0{\cdot}40$.

The gross power of the fall being equal to $P \times H$, that is to say to the weight P of water flowing per second multiplied by the difference H in the levels of the upper and lower reaches, the theoretical efficiency would be $\frac{Tu}{PH}$.

To obtain the actual efficiency this expression needs multiplying by a reduction factor of 0·85 or 0·90, which gives a total efficiency of about 35 per cent.

Further it is seen, that in utilising a portion of the kinetic energy of the water to produce a stationary wave, a change of level $(i' - i)$ is obtained such that the fall H is increased as much and becomes:

$$H' = H + (i' - i).$$

Actually then the work received by the motor is 0·35 PH′ instead of only 0·35 PH.

The vertical opening under the gate may vary from 8 inches as a minimum to 20 inches in seasons of greatest flood.

The volume q per foot of breadth per second is:

$$q = m \, . \, e \, . \, v = m \, . \, e \times \sqrt{2g\,(h \;-\; 0{\cdot}8e)},$$

the contraction coefficient m may vary from 0·74 to 0·80 as the inclination of the gate to the horizon passes from 70 to 45 degrees.

Letting l denote the breadth of the wheel, and Q the total volume flowing per second, which is necessarily known, the breadth is settled by the relation:

$$l = \frac{Q}{q}.$$

The diameter of undershot wheels lies between 11 and 23 feet according to the height of the fall.

The velocity V being given by the formula:

$$V = \sqrt{2g\,(h - 0{\cdot}8e)},$$

v may be deduced by the relation previously indicated,

$$\frac{v}{V} = 0{\cdot}40,$$

or: $$v = 0{\cdot}40\,V,$$

or more generally from the equation:

$$yV = y'v,$$

whence: $$v = \frac{y}{y'} \times V.$$

As regards the depth $y' = i = p$, this depends on the fraction of the buckets filled and consequently on the position of the lower edge of the gate. This fraction varies from $\frac{2}{3}$ to $\frac{4}{5}$.

The speed of rotation, or the number of turns per minute, is given as always by the relation:

$$n = \frac{60v}{2\pi R}.$$

The number N and the spacing E of the blades are calculated from the empirical formulae:

$$N = 4R,$$

and: $$E = \frac{2\pi R}{N} = \frac{\pi}{2} = 1{\cdot}57 \text{ ft. or 19 inches, say.}$$

In practice this spacing may vary from 18 to 24 inches without inconvenience.

As has been stated, by the artifice of the stationary wave, about half the height of the fall $h' = \frac{v^2}{2g}$ due to the speed v of the water on leaving the wheel, may be recovered. Supposing the head necessary to produce the speed V to be 3 feet, we have:

$$V = \sqrt{64 \times 3} = 8\sqrt{3} = 13{\cdot}85,$$

and: $$v = 0{\cdot}40V = 5{\cdot}54,$$

whence: $$\frac{v^2}{2g} = 0{\cdot}48,$$

and: $$\frac{1}{2} \cdot \frac{v^2}{2g} = 0{\cdot}24.$$

Here it is seen that 0·24 feet is gained in 3 feet so that the stationary wave increases the efficiency of the wheel by 8 per cent.

The undershot wheel is a high speed wheel; hence its dimensions are compact and the quantity of water dealt with relatively great; its construction and installation are extremely simple; it is economical therefore from these points of view. But its efficiency is low and much less than that of the side wheel; hence it does not enable good use to be made of a fall and is only suitable for simple installations, to drive machinery at high speed and when ample hydraulic power is available.

36. The Poncelet Wheel with Curved Blades. This wheel, as its name indicates, differs from the preceding one, in that the blades instead of being flat, are, in profile, arcs of circles. These curved blades lie between two flanges or

shrouds as in the bucket wheel, but there is no inner rim (fig. 27).

The water is fed in similarly to the case of the flat-bladed wheel as shown in fig. 27, but the wheel is not often immersed in the tail-race, the foot of the wheel being almost tangential to the surface N′.

The bed which was originally an inclined plane, was improved by Poncelet, the inventor of this wheel, so as to form a cylindrical bed.

The construction of this bed is as follows: let *ax* be the direction of a flat channel as settled by the condition of

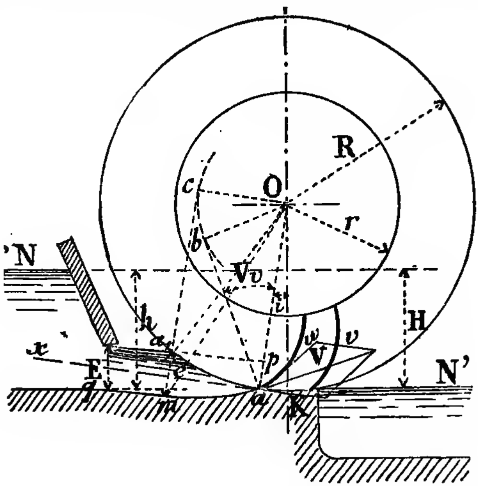

Fig. 27. Poncelet wheel.

sufficient slope, say about $\frac{1}{10}$, to maintain the flow over the bed at the speed given to the water in its passage under the gate. On *ax* erect the perpendicular O*a*, making with the vertical OK the angle *i* equal to the inclination of the straight bed to the horizon. Then from *a* draw the straight line *ab* making with O*a* an angle of 15 degrees equal to that which ought to be made between the directions of the velocities V and *v*.

As a result of this construction the line ab is perpendicular to V, while Oa is perpendicular to v whose direction is ax, the tangent to the circumference of the wheel at a.

This being done, the circle of radius ob is drawn about the centre O tangential to the line ab; on this circumference an arc bc is taken whose development equals the thickness $e = 0{\cdot}8\text{E}$ of the sheet of water and the tangent cm is drawn at the point c.

Then the arc am is drawn which is the development of the circle of radius Ob. In this manner a curved cylindrical floor is made which directs the sheet of water in curved stream-lines parallel to ma. Consequently all the different stream-lines cut the outer circumference of the wheel at a constant angle Vv, since the angle ca_1o, for example, is simply the angle bao, the apex of the triangle being displaced from a to a_1.

Under these circumstances, the relative velocity of the various stream-lines is tangential to the outer portion of the curved blade throughout the whole thickness of the sheet, whereas for the channel with the plane bed, the stream-lines would cut the circumference at different angles striking the blades on entering the wheel, either on the concave surface, or even below on the convex face, and causing shocks which would impair the efficiency of the wheel.

If the channel be straight, the profile of the buckets should be traced to suit the upper stream-line and not the lower one, in order to avoid shocks from the other stream-lines on the back of the blades, that is to say in the inverse sense to the rotation of the wheel. In this case the upper stream-line would be the only one entering the wheel without shock, the outline of the blade would be fixed at the extremity of the radius oa_1 and the point a_1 depending directly on the thickness E of the sheet of water, the radius R would also depend on this thickness. Consequently this radius should be proportional to the thickness E, and increase with it, because if the point a_1 be raised on the circumference, the centre O needs to be raised at the same time on the vertical OK, so that the angle a_1Op which is equal to Vv may remain constant, as it ought.

This is not the case, however, with the cylindrical bed, since the construction of the outline of the blade was made starting from the point a, so that whatever the thickness E of the sheet of water, the radius of the wheel is quite independent of it.

Thus for $E = 8$ inches say, and for the ordinary angle $Vv = 15$ degrees the radius R might be reduced to below 10 feet without disadvantage in the case of the cylindrical bed, whereas with the straight channel the radius ought to be 15 feet.

Such are the advantages resulting from the curved channel.

To complete the bed, the circular portion already dealt with is smoothly joined to the sill from m to q by the arc of a circle of large radius; lastly on leaving the point a the wheel runs in a circular channel K which clears the wheel by about half an inch.

In the curved-bladed wheels of the Poncelet system, the water does not act in the same manner as in the previous case. It is in virtue of the kinetic energy due to the head h of the water, which is practically the same as the height H of fall, that the water acts on the wheel.

Let V be the absolute velocity of the water on entering the buckets and V_1 the absolute speed at the outlet. The kinetic energy of the mass of water $\frac{P}{g}$ entering the wheel each second will be equal to $\frac{Pv^2}{2g}$; on leaving it will be only equal to $\frac{PV_1^2}{2g}$ and the difference:

$$Tu = \frac{PV^2 - PV_1^2}{2g} = \frac{P(V^2 - V_1^2)}{2g},$$

gives the useful energy received by the wheel and transformed into mechanical work.

Supposing now that $v = w = \frac{V}{2}$ which conditions correspond to the highest efficiency, as the foregoing discussion has shown, then the velocity triangle is isosceles, and the angle

Vw being 15 degrees, the angle vw will be twice this, or 30 degrees.

Under these conditions, the formula for the useful work reduces to:

$$Tu = 0{\cdot}933 \frac{PV^2}{2g},$$

and since $\frac{V^2}{2g}$ is practically equal to H, it follows that:

$$Tu = 0{\cdot}933P \times H.$$

That is to say the energy received by the wheel would be more than 90 per cent. of the gross fall.

But the conditions assumed can never be exactly realised, for the stream-lines do not all behave alike and eddies are caused in the buckets which occasion loss of energy. In practice, the efficiency of the Poncelet wheel, although considerably higher than for other undershot wheels, varies from 65 to 55 per cent. for falls of from 5 to 7 feet.

The head of water is very nearly equal to the height of the fall H, the sill of the channel being almost level with the tail-race surface. Whenever the tail-race level is variable, the angle i is increased so as to raise the sill, in order to avoid drowning of the orifice in time of flood.

The absolute velocity V is given by the same equation as in the case of the wheel with the flat float-boards.

According to the relations assumed above, one should have:

$$v = w = \frac{V}{2}.$$

But actually since the speeds v and w make a certain angle between themselves, which in this case is 30 degrees, V is the sum of the projections of these two velocities on its own direction, and consequently:

$$V = (w + v) \times 0{\cdot}966 = 2v \times 0{\cdot}966,$$

whence:

$$v = \frac{V}{1{\cdot}932} = 0{\cdot}53V.$$

The number of turns per minute is obtained from :

$$n = \frac{60v}{2\pi R}.$$

The depth of the bucket ring should be such that the water which runs up the length of the blade in virtue of its relative speed w, will not pass over the top of this blade; now the theoretical height to which it may reach is equal to $\frac{w^2}{2g}$. It is always advisable to give the bucket ring a greater depth and practically to obtain this depth the following relation is taken :

$$R - r = \frac{H}{3} + E.$$

Again the relation already established may be used, allowing a fractional filling between $\frac{1}{3}$ and $\frac{2}{7}$.

The spacing E of the blades may be nearly equal to 10 inches when they are of sheet-iron, when of wood it may be taken up to 18 inches as a maximum.

Of course the number of blades is given by :

$$N = \frac{2\pi R}{E}.$$

It has been stated that the efficiency of the Poncelet wheel may reach 65 per cent.; the maximum efficiency has been observed to occur when the wheel is immersed 5 inches in the tail-race, diminishing for greater immersion.

The Poncelet wheel, like other undershot wheels, has a relatively high speed; its efficiency is almost independent of the flow and also of the speed when the curved approach channel is used. Moreover this speed is constant in spite of variations of level within wide limits to which the head-reach may be subject.

37. Floating wheels with flat blades. This type of wheel is the most simple imaginable. The construction is the same as for undershot wheels with flat blades, but the wheel is simply erected in the current, without any dam or sluices (fig. 28).

The flat blades are mounted on inner circles but without being enclosed by shrouds, so that the water may freely enter the spaces.

The depth of the current and its breadth must be very large as compared with the size of the blades, so that the latter dip only into the upper portion of the current and do not reach near the bed of the river where the water is subject to eddies.

The blades, thus plunging into the current, receive a part of the kinetic energy of the stream-lines which may be supposed to all possess the same velocity over the depth to which the blades reach.

Obviously the mass of water acting on the wheel each second, depends on the area of the paddle and on the speed V of the current. If therefore P represent the weight of this mass, S the area of the paddle, and K be a coefficient which allows for the fact that the spaces between consecutive blades are not completely filled with water, then:

$$P = K \times d \times S \times V,$$

where d is the weight of a cubic foot of water, namely 62·4 lbs. Generally K has the value 0·80.

This equation cannot always be correct, for it takes no account of the speed of the wheel; now it is evident that each time a paddle reaches the vertical position, a volume of water equal to that filling the space considered is delivered. It would therefore be more accurate to make use of the relation previously established:

$$P = d \times 0{\cdot}9l(R^2 - r^2) \times \frac{v}{2R},$$

in which R and r are the radii of the outer perimeter and the circle tangential to the surface of the water in the inside of the space, l the breadth of the wheel and v its circumferential velocity.

The floating wheel thus erected receives therefore the impulse F of the mass of water striking the paddles each second, and according to the law of momentum this impulse is expressed by the change of this momentum per second.

The mass $\frac{P}{g}$ possesses the velocity V of the current at the instant it enters the wheel; afterwards it travels with the speed v of the wheel with which it moves; at this instant its momentum has changed from the value $\frac{P}{g} \times V$ to the value $\frac{P}{g} \times v$, so that the change is:

$$\frac{P}{g} \times V - \frac{P}{g} v = \frac{P}{g} \times (V - v).$$

Hence:

$$F = \frac{P}{g} (V - v).$$

This then is the value of the impulse produced by the moving mass of liquid on the hydraulic motor. Further, as the work T is always the product of force by displacement in the direction of the force, it follows that:

$$T = F \times v = \frac{P \times (V - v)}{g} \times v.$$

Which may be written:

$$T = \frac{P}{g} \times (V - v) \times v.$$

Now it may be shown that the product $(V - v) \times v$ is a maximum when the two factors $(V - v)$ and v are equal, and when therefore:

$$V - v = v,$$

whence:

$$2v = V \quad \text{or} \quad v = \frac{V}{2}.$$

If this condition be satisfied, the value of the maximum work is given by:

$$T = \frac{P}{g} \times \frac{V^2}{4} = \frac{1}{2} \cdot \frac{PV^2}{2g}.$$

Now $\frac{PV^2}{2g}$ is the kinetic energy of the water and it is seen that the energy received by the wheel is as a maximum only half this energy. Consequently, the efficiency of the wheel with which we are now dealing can never exceed 50 per cent. theoretically; actually it never exceeds 40 per cent.

To settle the chief dimensions of the floating wheel, one should know first the velocity V of the current which can easily be measured. Next one might chose some efficiency coefficient, say 0·45, for example, and then deduce P by the preceding relation:

$$T = 0{\cdot}45 \, . \, \frac{PV^2}{2g},$$

whence:

$$P = \frac{2g}{0{\cdot}45V^2} \times T.$$

The speed v of the wheel follows from the value of V for it must be such that:

$$\frac{P}{g} \times v \times (V - v) = 0{\cdot}45 \, \frac{PV^2}{2g},$$

from which one has:

$$2v \times (V - v) = 0{\cdot}45V^2.$$

The value of v may then be calculated by successive approximations; it is seen that according to the chosen hypothesis as regards efficiency, $v = 0{\cdot}345V$ satisfies the equation.

The diameter 2R is fixed according to what is convenient in each particular case; it is generally between 13 and 20 feet. The depth of the paddles $(R - r)$ varies between $\frac{R}{4}$ and $\frac{R}{5}$, and the distance between them is from 20 to 40 inches.

The paddles ought always to dip into the river to the same extent; this condition is naturally fulfilled when the wheel is mounted on a boat which follows the fluctuations in surface level; but when driving stationary machinery, it is necessary to adopt suitable devices to obtain this required condition of constant depth of paddle immersion.

Figure 28 shows the principle of these installations. The wheel is supported by a kind of hinged parallelogram, two sides of which are made by two cross-beams of cast-iron a, only one of which shows in the drawing; these cross-beams can turn about the fixed axle b, at the other extremity they are hung from rods c, attached to a raising mechanism, by

which the wheel may be raised or lowered to follow the variations in the level of the current's surface.

The rotatory motion of the wheel is transmitted to the fixed axle by a toothed wheel *d* and a pinion *f* mounted on this axle.

This wheel as thus erected rather merits the name of *hanging wheel*; but Colladon, a Swiss engineer, designed an arrangement which was actually a floating wheel, the blades

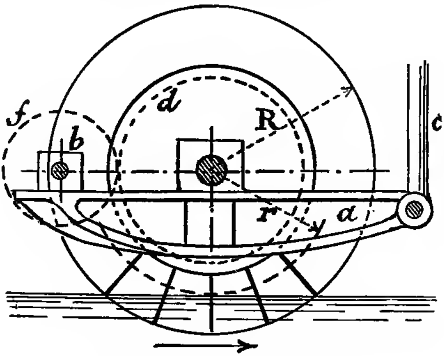

Fig. 28. Floating wheel.

being mounted on a water-tight cylindrical cask which rolled on the surface of the water, automatically rising and falling with the surface level of the river.

It is evident that whichever arrangement is adopted, the floating or hanging wheel is especially noticeable for the simplicity of its construction and installation. But the efficiency is low and the system cannot be used save in particular cases and only for relatively insignificant powers.

TURBINES.

38. Outward-flow or Fourneyron Turbine. We have already classified turbines both as regards the manner in which the water flows through them (outward-flow, inward-flow, axial and mixed-flow turbines), and also according to

the manner in which the water acts on the moving blades (impulse or free-deviation turbines, reaction or pressure turbines, and turbines limited to no reaction) (§ 28).

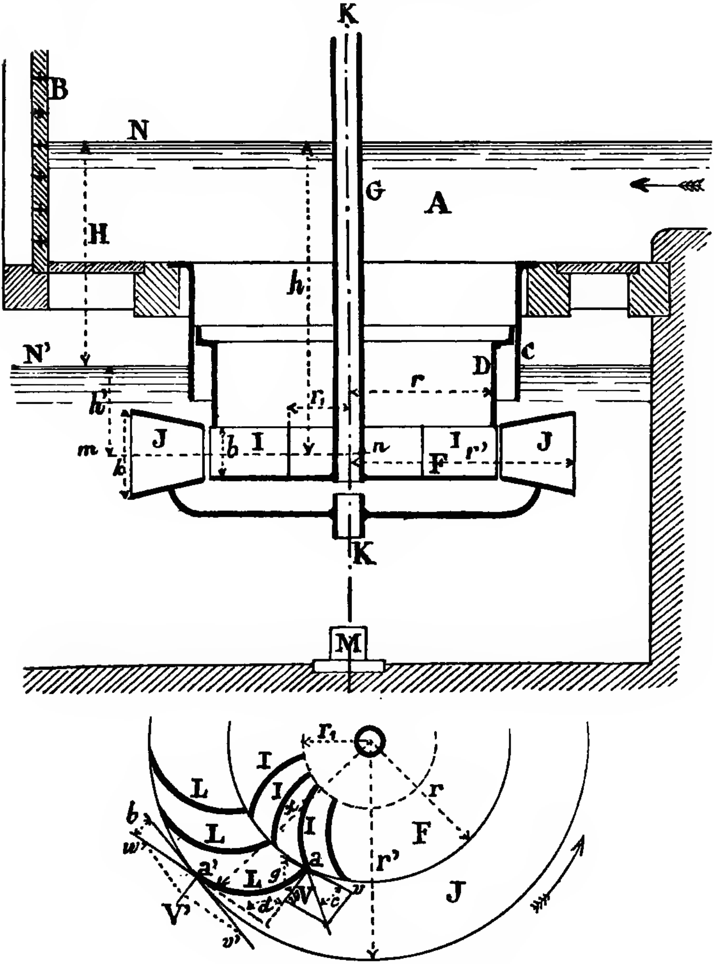

Fig. 29. Outward-flow turbine.

We will first examine the outward-flow turbine both with and without reaction.

Fig. 29 shows the general arrangement of the installation of such a turbine.

The bottom of the chamber A, where the feed-canal ends, is provided with a cast-iron tube C. Inside this tube a cylinder D with a water-tight joint can slide up or down, this cylinder forms the circular gate whose function it is to close to a greater or less extent the outlet of the guide ring channels I.

The guide blades are of cast-iron attached to the fixed plate F, supported at the end of a tubular sheath G, which is in turn bolted to the floor which covers the water chamber. The directing channels have a uniform depth b throughout their length. The water fills these guide channels and escapes horizontally through the periphery of the fixed crown taking a direction tangential to the last portion of the guiding partitions.

The moving ring which constitutes the motor portion of the machine, is concentric with the fixed crown. It is formed by two cheeks in the form of truncated cones with their small ends nearest each other. Between these two cheeks the curved blades L are situated, the curvature of these is in the inverse sense to that of the directing blades. It is obvious therefore that the water cannot escape from the latter into the tail-race without tending to turn the moveable crown.

This latter is mounted on a vertical axle K which passes through the sheath G and turns in the footstep bearing M, built in to the bed of the tail-race. The work given to the moving crown by the water is transmitted through this axle.

The lower figure is a horizontal section through the fixed and moving crowns on the plane mn. The water moving in this mean plane possesses an absolute velocity V at the outlet a of the fixed crown. At this point the water enters the moving crown; but this latter turns with a tangential velocity v at the aforesaid point a. The stream-lines may be considered as participating in this turning speed of the wheel, and

further as gliding over the curved moving blade with a relative speed w.

A parallelogram may be drawn connecting these three velocities just as in the case of forces and according to the same rule.

The direction of the relative velocity w which is thus determined, serves to fix the direction of the first portion of the moving blade. This blade should therefore be tangential to the speed w at a, so that the stream-lines travelling in this direction may enter the moving crown without shock.

The three speeds being thus related by the sides of a parallelogram or of a triangle having the same vectors for sides, it will be perceived that the speed v of the moving crown depends not only on the value of V, but also on the greater or less degree of inclination of the sides of the triangle.

The angle between v and V depends on the direction of V, that is to say on the direction of the outer portion of the fixed blade, which may join the circumference at a more or less acute angle. It is also necessary to take into account the angle g made by the first portion of the moving vane, or the inclination of the speed w to the speed v.

Instead of considering these angles, one may consider the inclinations of V to w and of w to v, which depend directly on them. Let us use I and I_1 to denote respectively these inclinations.

From this we can write the fundamental connection deduced from the geometrical properties of the triangle under consideration:

$$\frac{v}{V} = \frac{I}{I_1} \times \frac{\sqrt{1 + I_1^2}}{\sqrt{1 + I^2}}.$$

The water enters the moving crown with the relative speed w.

On applying the law of growth of kinetic energy in the sheet of current due to variations of the relative speeds w and

w' which this sheet possesses, in the entering and leaving sections a and a', the following equation is obtained:

$$* \quad w'^2 = w^2 + v'^2 - v^2 + 2g\left(\frac{p_0}{d} - \frac{p_1}{d}\right).$$

In this equation:

w' is the required relative speed at the outlet of the moving crown;

v' is the speed of the crown at the outside circumference of radius r';

v is the speed of the same crown and the inner circumference of radius r;

p_0 is the hydrostatic pressure at a per unit area of section of the liquid sheet;

p_1 is the hydrostatic pressure on the outlet section at a' which is exerted from without inwards, in the contrary direction to the previous one.

Finally d is the weight of the liquid per unit volume, that is per cubic foot.

Now, evidently:

$$p_1 = p_a + dh',$$

h' being the depth of tail-race water above the mid-horizontal plane mn.

Putting this value of p_1 in the previous expression, this may be written:

$$w'^2 = w^2 + v'^2 - v^2 + 2g\left(\frac{p_0 - p_a}{d} - h'\right);$$

v and w are known from a foregoing equation; as for v' this is deduced from the obvious relation:

$$\frac{v'}{v} = \frac{r'}{r}, \qquad \text{whence:} \quad v' = v \times \frac{r'}{r}.$$

We have supposed the speed V of the water at the outlet of the guide crown to be known. This is easily determined by writing that the head generating this velocity is equal to

* NOTE BY TRANSLATOR. The simple Bernoulli formula is here modified by the action of centrifugal force.

the sum of the heads corresponding to the pressures acting on the two faces, outer and inner, of the section of the liquid sheet at *a*.

The inner pressure corresponds to heads $\frac{p_a}{d} + h$, due to the atmospheric pressure p_a and to the column of water above the mid-plane *mn* of the moving crown; the outer pressure exerted in the opposite direction corresponds to $\left(-\frac{p_0}{d}\right)$.

Hence:
$$\frac{V^2}{2g} = \frac{p_a}{d} + h - \frac{p_0}{d}.$$

Also the triangle of velocities at *a* gives the relation:

$$w^2 = V^2 + v^2 - 2Vv\frac{1}{\sqrt{1+i^2}},$$

where *i* denotes the inclination of the direction of the velocity V to the tangential speed *v* at *a*.

At the outlet also a triangle of velocities may be constructed which will in the same way give the relation:

$$V'^2 = w'^2 + v'^2 - 2w'v'\frac{1}{\sqrt{1+i_1^2}},$$

where i_1 denotes the inclination of the relative speed w' to the circumferential speed v' at a'.

The water on leaving the turbine with an absolute velocity V′, bears with it energy equal to:

$$\tfrac{1}{2}mV'^2 = \frac{PV'^2}{2g},$$

which is unavailable for use.

The gross energy of the fall being equal to P × H, the product of the weight of water delivered per second by the height of the fall, or the difference in level H between the head and tail-races, the useful work is given theoretically by:

$$Tu = PH - \frac{PV'^2}{2g} = P\left(H - \frac{V'^2}{2g}\right).$$

Therefore the efficiency, which is the ratio of the useful power to the total power supplied, has as its value:

$$E = \frac{PH - \frac{PV'^2}{2g}}{PH} = 1 - \frac{V'^2}{2gH}.$$

This expression proves that the efficiency is better in proportion as V' is smaller; it would be equal to unity if V' were zero.

Now we have seen that:

$$V'^2 = w'^2 + v'^2 - 2w'v' \frac{1}{\sqrt{1 + i_1^2}}.$$

Obviously if the inclination i_1 of the last element of the moving vane to the speed v' were zero, in other words if the blade became tangential to the circumference, the above expression would become:

$$V'^2 = w'^2 + v'^2 - 2w'v'.$$

And if further it be supposed that $w' = v'$, it would finally follow that:

$$V'^2 = 2w'^2 - 2w'^2 = 0.$$

But the former condition can never be realised, for then the vanes would join one another and there would be no outlet for the water. For this reason one must be content to make the angle b small and comprised between 20 and 30 degrees.

As for the condition $w' = v'$, this can always be satisfied if the value of v' is sufficient for the turbine to deliver all the water supplied by the waterfall. In what follows therefore, we will assume that equality of relative speed and peripheral speed is always realised at the outlet of the moving crown.

The above relations hold generally for all outward-flow turbines whether impulse or reaction turbines.

But the particular characteristics of each of these different methods of working turbines introduce important modifications into the general formulae, which it is now advisable to consider.

Let us first of all make use of the assumed relation $w' = v'$, which when introduced into the preceding formulae will allow them to be simplified to some extent.

It should be observed at the outset, that the fundamental equation is that which gives the speed V of the water at the outlet of the guide crown. This speed depends on the height of the available fall and forms the chief factor of the energy received and used by the turbine. This formula is as we know:

$$\frac{V^2}{2g} = \frac{p_a}{d} + h - \frac{p_0}{d}.$$

Now everything in the second member of this equation is known save the back-pressure p_0, which is exerted inwards from without at the point a of the guide vane.

It is necessary therefore to find the value of $\frac{p_0}{d}$, and this value possesses a special interest because it distinguishes between the impulse and the reaction method of working.

For this purpose let it be noticed that in the above formula h may be replaced by $H + h'$.

We may also make use of the expression for w'^2 which gives the relative speed at a', and which becomes by the simplification resulting from the condition $w' = v'$:

$$0 = w^2 - v^2 + 2g\left(\frac{p_0 - p_a}{d} - h'\right).$$

Adding this equation to the preceding one, member for member, after having put this latter in a similar form:

$$V^2 = 2g\left(\frac{p_0 - p_a}{d} + h\right),$$

and recalling that $h - h' = H$, we obtain the relation:

$$V^2 = w^2 - v^2 + 2gH.$$

Replacing w^2 by its value given by the relation deduced from the triangle of velocities at a, it will become:

$$V = \frac{gH}{v} \times \sqrt{1 + i^2}.$$

Now,
$$w' = v' = v\,\frac{r'}{r},$$
whence:
$$v = w'\,\frac{r}{r'}.$$

Substituting this value in the expression for V, it becomes:

$$V = \frac{gH}{w'} \times \frac{r'}{r} \times \sqrt{1 + i^2}.$$

Therefore V can be expressed solely in terms of the principal dimensions of the turbine, if the relative speed w' in the moving crown can itself be expressed in terms of these dimensions. Now the required connection can be obtained from consideration of the volume of flow through the turbine.

Obviously the volumes per second flowing through the fixed and moving crowns are equal.

To estimate the volume flowing at the outlet from the guides of the fixed crown, it is necessary to consider the normal section of the sheet of water whose stream-lines are parallel to the direction V. We will assume, as a sufficiently near approximation in practice, that the tangent at a, for example, coincides with the arc l' embraced by two consecutive vanes.

The horizontal dimension l of the plane section of the sheet of water normal to the flow, forms with the tangent v at a an angle complementary to that made between the directions of the speeds v and V, so that if i denotes the inclination corresponding to this latter angle, then:

$$l = l' \times \frac{i}{\sqrt{1 + i^2}}.$$

The sum of the horizontal dimensions of the outlet sections is equal to the above expression multiplied by the number of spaces, but the thickness of the vanes must be taken into account. If, say, the thickness takes up 15 per cent. of the circumference, the available outlet will not be more than 0·85 l horizontally. Therefore the previous result must be multiplied by some coefficient K such as 0·85, which will always be less than unity.

The sum of all the widths l' is equal to $2\pi r$ and the available outlet in the horizontal dimension will be in consequence:

$$K \times L = 2\pi r \times \frac{i}{\sqrt{1 + i^2}} \times K.$$

The vertical dimension of the orifices being equal to b, the total section of the current through the orifices of the guide-crown is:

$$S = 2\pi r \times b \times \frac{i}{\sqrt{1 + i^2}} = K,$$

and the volume of flow per second:

$$Q' = 2\pi r \times b \times V \times \frac{i}{\sqrt{1 + i^2}} \times K,$$

since the water flows through the orifices in question with speed V.

But this formula would still give too great a value to the flow, for no account has been taken of the contraction which takes place in the channels. It will be further necessary therefore to multiply this result by a contraction coefficient m which varies from 0·85 to 1, according to the form and condition of the vanes. Finally then:

$$Q' = 2\pi r \times b \times V \times \frac{i}{\sqrt{1 + i^2}} \times K \times m.$$

The discharge from the outlet orifices of the moving crown may be similarly calculated and its value will be obtained on replacing V in the above formula by w', for w' is the speed with which the water passes across the section normal to the vanes. In addition, the inclination i must be replaced by i_1 the inclination of the speed w' to the peripheral speed v', and the height b by the height b'.

Then:

$$Q'' = 2\pi r' \times b' \times w' \times \frac{i_1}{\sqrt{1 + i_1^2}} \times K \times m,$$

assuming for simplicity that the coefficients K and m are the same in the two cases.

Eliminating the common factors:

$$r \times b \times V \times \frac{i}{\sqrt{1+i^2}} = r' \times b' \times w' \times \frac{i_1}{\sqrt{1+i_1^2}},$$

whence:

$$\frac{V}{w'} = \frac{r' \times b'}{r \times b} \times \frac{i_1}{i} \times \frac{\sqrt{1+i^2}}{\sqrt{1+i_1^2}}.$$

Moreover, if the two members of the equation which it was desired to transform be multiplied by V, it becomes:

$$V^2 = \frac{V}{w'} \times gH \times \frac{r'}{r} \times \sqrt{1+i^2}.$$

It only remains to replace $\frac{V}{w'}$ in this by the above value to obtain the desired result, that is:

$$V^2 = gH \times \frac{r'^2}{r^2} \times \frac{b'}{b} \times \frac{i_1}{i} \times \frac{(1+i^2)}{\sqrt{1+i_1^2}}.$$

The foregoing calculations give all the data required to obtain the value of the back-pressure p_0 in terms of the dimensions of the turbine. In fact from the fundamental relation:

$$\frac{p_0}{d} = \frac{p_a}{d} + h - \frac{V^2}{2g} = \frac{p_a}{d} + H + h' - \frac{V^2}{2g},$$

since $h = H + h'$.

Replacing V^2 in this equation by the value found above:

$$\frac{p_0}{d} = \frac{p_a}{d} + h' + H \times \left(1 - \frac{b'}{b} \times \frac{r'^2}{r^2} \times \frac{i_1}{i} \times \frac{(1+i^2)}{2\sqrt{1+i_1^2}}\right).$$

From which it is seen that the hydrostatic pressure $\frac{p_0}{d}$ is equal to the external pressure $\frac{p_1}{d} = \left(\frac{p_a}{d} + h'\right)$ increased by an amount, which, as a matter of fact, may need to be added or subtracted according to whether the negative portion in the brackets is less or greater than 1; if this same term is equal to 1, the quantity in brackets becomes zero and the third

quantity disappears; there are then three cases to be considered according to whether:

$$\frac{p_0}{d} > \frac{p_a}{d} + h';$$

$$\frac{p_0}{d} < \frac{p_a}{d} + h';$$

or

$$\frac{p_0}{d} = \frac{p_a}{d} + h'.$$

In the first case, the outward pressure at a being greater than the inward pressure, water will escape through the mechanical clearance existing between the two crowns; the reverse phenomenon takes place in the second case, and water will enter the moving crown from the tail-race through the aforesaid clearance. In both of these cases loss of energy results.

If on the other hand equality between the external and internal pressure is realised, no loss of this kind is produced. It will be seen later that this is the essential condition for the working of the free deviation turbine.

The relation expressing equality between outward and inward pressure at the outlet of the moving crown, namely:

$$\frac{p_0}{d} = \frac{p_a}{d} + h$$

becomes simplified again when this crown is entirely out of the water, in this case h' is zero, and:

$$\frac{p_0}{d} = \frac{p_a}{d}.$$

For this condition to be satisfied, it is not sufficient for the turbine to be completely out of the water, but atmospheric pressure must be able to act on the free surface of the water-jets.

Therefore there must in the first place be a free surface, and to this end the sections of the passages in the moving crown are larger than would be strictly necessary to allow the

water coming from the directing vanes to pass. By this means the whole capacity of the moving crown is not taken up, the water is not in contact with the convex surface of each vane but glides over the opposite concave face, as if the motion were taking place in an open channel.

Lastly, in order that atmospheric pressure may be exerted on the surface of the sheet of water, holes or *vents* are arranged in the upper cheek of the moving crown, one behind each vane, their breadth being less than the space between the free surface of the liquid sheet and the convex face of the vane.

Obviously under these conditions the sheet of water no longer exerts any reaction on the vanes, as it would if the spaces were entirely filled with water, and its action is due solely to the kinetic energy of the liquid jets which are deviated from their original direction. There is therefore good reason for giving to this mode of working the name *free deviation.*

Actually therefore it is the relation $p_0 = p_a$ which characterises this kind of turbine.

It should further be noted that it is impossible to have free deviation, such as we have just defined, with a drowned turbine; because this circumstance is incompatible with the presence of vents designed to insure that the flow shall take place in the presence and under the action of atmospheric pressure.

Nevertheless a turbine may work drowned without making use of any reaction. For this purpose it is sufficient if the sections of the passages are such that the flow is unaffected by either vacuum or pressure, and the general condition is satisfied that:

$$\frac{p_0}{d} = \frac{p_a}{d} + h',$$

where h' is the depth to which the turbine is drowned.

Let us assume in what follows that we have in mind the turbine without reaction and that the above condition is always satisfied.

This relation may be written :

$$\frac{p_0 - p_a}{d} = h'.$$

Putting this in the formula already established :

$$0 = w^2 - v^2 + 2g\left(\frac{p_0 - p_a}{d} - h'\right),$$

it simplifies to : $$0 = w^2 - v^2,$$

or : $$w = v.$$

It is seen then that this relation always results from the general condition stated, whether h' = nothing or not. Therefore it always characterises working without reaction. This condition of equality of the two speeds being realised, the turbine is a free-deviation one if undrowned, and a turbine *limited to no reaction* if drowned.

A turbine can never be truly said to be a free-deviation turbine unless all the conditions and arrangements enumerated above are satisfied, both as regards the free surface of the liquid and the provision of vents in the upper cheek of the moving crown.

The above condition makes the triangle of velocities at the point a isosceles (fig. 29) ; it would become equilateral if the angle between the directions of the two speeds v and V were equal to 60 degrees, in which case we should have :

$$w = v = V.$$

Again putting the relation characteristic of non-reaction in the formula giving the value of V, namely :

$$\frac{V^2}{2g} = \frac{p_a}{d} - \frac{p_0}{d} + h,$$

it becomes : $$\frac{V^2}{2g} = h - h' = H,$$

whence : $$V = \sqrt{2gH}.$$

That is to say the speed of the water on leaving the guide channels is equal to that corresponding to the height of the

fall. This circumstance only follows when the conditions previously expressed are realised. The above relation is therefore also characteristic of working without reaction, whether drowned or not.

Further, as has been explained further back, turbines may be free deviation whether installed above high-water level in the tail-race, or just the same if below it, if *hydropneumatisation*, of which we shall speak presently, be resorted to.

In the case of reaction working, the velocity V is given by the general relation obtained before:

$$V = \frac{gH}{v} \times \sqrt{1 + i^2}.$$

The following relation has also been established previously:

$$\frac{v}{V} = \frac{I}{I_1} \times \frac{\sqrt{1 + I_1^2}}{\sqrt{1 + I^2}}.$$

Combining these two, the general formula is easily obtained in the form:

$$V^2 = gH \times \frac{I_1}{I} \times \frac{\sqrt{1 + I^2} \times \sqrt{1 + i^2}}{\sqrt{1 + I_1^2}}.$$

Now in non-reaction working, the velocity w equals the velocity v, and the triangle of velocities at a is isosceles, that is to say the two velocities are equally inclined to the direction of the speed V, hence $I = i$.

As a result the formula is simplified, and may be written:

$$V^2 = gH \times \frac{(1 + i^2)}{i} \times \frac{I_1}{\sqrt{1 + I_1^2}}.$$

But since the angle g has for its supplement twice the angle which v makes with V and which corresponds to the inclination i, the relation may be established that:

$$\frac{I_1}{\sqrt{1 + I_1^2}} = \frac{2i}{(1 + i^2)},$$

which when brought into the above formula, brings us back to the expression:

$$V^2 = 2gH,$$

or :

$$V = \sqrt{2gH}.$$

This relation follows from the hypothesis $v = w$; if v is greater than w, as in reaction turbines, one must use the general formula in which the factors, in terms of the inclinations, have a value less than 2, so that V is smaller than in turbines without reaction and therefore :

$$V < \sqrt{2gH}.$$

The ratio of the effective speed at the outlet of the guide ring to the theoretical speed $\sqrt{2gH}$ which applies to turbines working without reaction, is called the degree of reaction :

$$K = \frac{V}{\sqrt{2gH}}.$$

The equality of the angles or the inclinations I and i also enables the value of the ratio $\frac{v}{V}$ to be simplified as follows :

$$\frac{v}{V} = \frac{\sqrt{1 + i^2}}{2}.$$

Obviously this ratio depends solely upon the inclination of the speed V at the point a.

As the angle corresponding to the inclination varies from 0 to 90 degrees, this ratio goes on increasing from 0·50 to a theoretically infinite value.

This angle can never be zero, for then the last element of the vane of the directing crown would coincide with the circumference of this crown and the area of the outlet section would be nothing.

Neither can it reach 90 degrees, because then the vanes would be in line with the radii and be in consequence entirely useless.

However the ratio $\frac{v}{V}$ allows turbines to be classed from the point of view of speed. By slow-speed turbines are meant those for which this ratio is less than 0·60; for high-speed turbines this ratio is near unity.

It has been seen that the turbine efficiency is expressed by:

$$E = \frac{H - \dfrac{V'^2}{2g}}{H}.$$

This relation may be simplified and written:

$$E = 1 - \frac{V'^2}{2gH}.$$

It has been observed already that to obtain maximum efficiency, two conditions must be satisfied. The first is that w' must be made equal to v'; as, in every case, it is this relative speed w' which decides the flow through the moving crown, it is necessary that the value of v' should be sufficient to ensure the total flow of the fall if the above condition is to be realised.

As to the second condition, which can never be exactly attained in practice, it is always approached very closely by making the angle between the directions of the two speeds as small as the requirements of the flow will permit. In practice this angle lies between 20 and 30 degrees.

It may be shown further that the efficiency also depends on the inclinations of the speeds v and V to one another. Briefly, the efficiency is greater as the angles made between the speeds w' and v' on the one hand, and the speeds v and V on the other hand, are smaller.

But we have seen already that on the inclination of v to V depends the speed of the turbine, low speeds corresponding to small inclinations and *vice versâ*.

Hence low-speed turbines have the greater efficiency.

Moreover, the equation giving the condition relating to free deviation, namely:

$$\frac{b'}{b} \times \frac{r'^2}{r^2} \times \frac{i_1}{i} \times \frac{(1 + i^2)}{2\sqrt{1 + i_1^2}} = 1,$$

may be written:

$$\frac{b'}{b} = \frac{r^2 \times i \times 2\sqrt{1 + i_1^2}}{r'^2 \times i_1 \times (1 + i^2)}.$$

And as V', which enters into the expression for the efficiency, necessarily depends on the dimensions b and b' of the inlet and outlet orifices respectively, and consequently on r' and r according to this relation, obviously V' will be smaller and the efficiency greater in proportion as the ratio $\frac{r'^2}{r^2}$ approaches unity.

Thus it is of advantage to make r' as little different as possible from r. But in practice, one is compelled to take values for r' and r differing considerably in order to give sufficient area to the moving vane, and generally it may be assumed that:

$$r' = 1{\cdot}5r \text{ to } 2r.$$

The above formulae give the theoretical efficiency; to obtain the actual or practical efficiency the former must be multiplied by a coefficient less than unity and lying between 0·80 and 0·90.

In short, it may be assumed that according to the angles chosen for the inclination of the vanes to the perimeter, and according to the ratio of the outer and inner radii, the efficiencies vary from 0·70 to 0·75 for low-speed turbines and from 0·58 to 0·62 for high-speed turbines.

The expression for the output of the turbine:

$$T = P \times \left(H - \frac{V'^2}{2g}\right),$$

contains the total height H of the fall. This expression assumes therefore that the moving crown is completely submerged in the water of the tail-race.

In fact the figure already given shows the median plane mn of this crown at a depth h' below the surface of the water in the tail-race. If, instead of this being the case, the median plane were situated at a height h'' above the level N', it is evident that the fall would be reduced by this same amount and the work given out would be reduced to:

$$T = P \times \left(H - h'' - \frac{V'^2}{2g}\right),$$

and the efficiency would be:

$$E = 1 - \frac{V'^2}{2gH} - \frac{h''}{H}.$$

It follows that the loss of useful work and efficiency is greater as h'' becomes a larger proportion of H. This loss is always of more relative importance for low falls than for high, therefore it is advantageous to drown the turbines in the former case, placing them below low-water level.

To vary the power of the turbine in order to suit the requirements of the installation, it is necessary to vary one of the factors of the energy, and it is on the flow P that the regulation is made.

In the Fourneyron turbine which is now under consideration, this variation is obtained by the circular gate which is interposed in the clearance left between the two crowns. But when the gate is only partially raised, the jets of water thus throttled on entering the moving crown, expand in the larger spaces of this crown and give rise to eddies and consequently considerable loss of energy.

It follows that the maximum efficiency at full bore constantly diminishes with the lowering of the gate corresponding to reduction of work. This diminution in efficiency may reach as much as 50 per cent.

To remedy this disadvantage, Girard, a French engineer, introduced the *partial admission* system represented in fig. 30. This was about the year 1850.

In this arrangement, there are only fixed guide-blades in two diametrically opposite quadrants, and these guide-blades are twice as numerous as are the vanes in the moving crown in the corresponding arc. The other two quarters of the circumference are occupied by a cylindrical partition formed by a downward prolongation of the tube D.

Behind the guide-blades is arranged a circular gate, also occupying the space of a quadrant, and capable of being turned through a quarter of a revolution, thus opening successively the channels numbered 1 to 10 in the direction of the moving crown.

In this way a greater or less number of orifices are opened to suit the demand and variations in the flow, instead of reducing the section of each by a proportionate amount, and in consequence eddies are avoided and the efficiency of the turbine is almost independent of the flow.

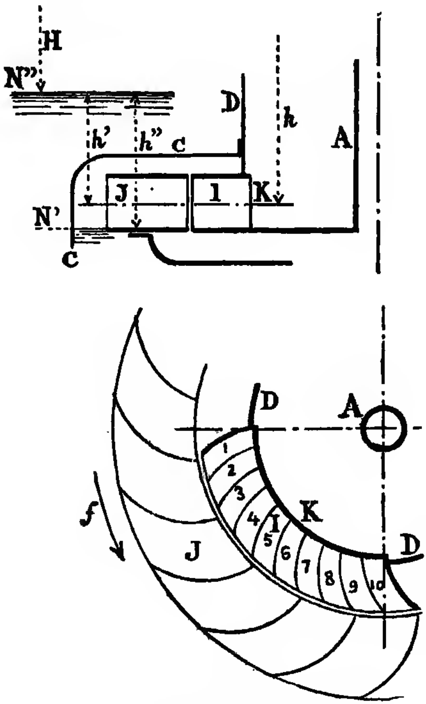

Fig. 30. Girard's partial admission.

39. Hydropneumatisation. Partial admission applied to the Fourneyron turbine requires that the moving crown turn in the air and not in the water. In fact, if this crown is submerged, half the blades only are fed through the two quadrants of guide-blades, while the other half are filled with

tail-race water which is thus held in the turbine in a state of relative rest, and the water entering the turbine from the guide-channels strikes the water at relative rest with considerable shock producing a relatively large loss of energy.

For this reason it is necessary to install the moving crown above the tail-race level, with consequent loss of fall as explained above.

Girard further remedied this disadvantage by attaching to the tube D a dome or bell C which encased the moving crown and dipped a little below its lower edge. By means of a small pump driven by the turbine itself, air is forced into the dome under sufficient pressure to prevent drowning of the turbine. To this device Girard gave the somewhat cumbrous name *hydropneumatisation.*

Hence hydropneumatisation allows the median plane of the turbine to be placed below the tail-race level.

Let h' be the depth at which this median plane is situated below the tail-race surface, and let h'' be the depth of the water surface inside the dome after the lowering of level produced by the compressed air (fig. 30).

This latter depth h'' represents the height of a column of water in equilibrium with the pressure of the compressed air, so that the pressure exerted at the entrance of the moving vanes under the bell, has for its value:

$$\frac{p_0}{d}=\frac{p_a}{d}+h'',$$

whereas if the turbine were drowned, this expression would be:

$$\frac{p_0}{d}=\frac{p_a}{d}+h'.$$

This will be recalled as the condition for free deviation working when there is neither forcing out nor sucking in of water through the clearance existing between the two crowns.

Putting the first value of $\frac{p_0}{d}$ in the fundamental equation:

$$\frac{V^2}{2g}=\frac{p_a}{d}+h-\frac{p_0}{d},$$

this becomes:

$$\frac{V^2}{2g} = \frac{p_a}{d} + h - \frac{p_a}{d} - h''.$$

And since:

$$h = H + h',$$

it follows that:

$$\frac{V^2}{2g} = H - (h'' - h'),$$

whence: $$V = \sqrt{2g\,[H - (h'' - h')]}.$$

When the turbine was drowned, without hydropneumatisation, we had: $$V = \sqrt{2gH}.$$

Obviously then this device has the effect of causing a loss of fall represented by the height $(h'' - h')$, that is the half-height of the turbine crowns.

The result therefore is the same as if the turbine had been placed above the tail-race level N″, in a relatively similar position to that which it occupies in relation to the level N′.

Hence hydropneumatisation would give no advantage, if the tail-race level were constant; but when this latter is capable of rising and drowning the turbine, the inconvenience which would result may be thus guarded against.

40. Calculations for an Outward-flow Turbine. The various relations set forth thus far, enable the principal dimensions of a Fourneyron turbine with a vertical axle, to be calculated.

Referring to the drawing of this turbine (fig. 29), without the hydropneumatisation device, it will be seen that the radius r of the tube is just the same as that of the outer rim of the guide-crown.

This radius is determined by the condition that the velocity u of the water flowing downwards in the tube, shall not exceed 3 feet per second.

Therefore writing that the efflux Q is equal to the horizontal section of the tube multiplied by the speed:

$$Q = \pi r^2 \times u,$$

it follows at once that:

$$r = \sqrt{\frac{Q}{\pi u}}.$$

When the turbine is provided with the device for hydropneumatisation, the radius r'' of the tube is that of the inner circle of the guide-crown. The outer radius r of the guide-crown is given by the relation:

$$r = 1{\cdot}20r'' \text{ to } 1{\cdot}50r''.$$

The water is discharged from the whole circumference of the guide-crown through an orifice of height b and whose base would be equal to the development of the circumference $2\pi r$, were it not necessary to take into consideration the thickness of the vanes which take up a certain amount of the section of the current.

These guide-vanes have thicknesses varying from $\frac{1}{12}$ to $\frac{3}{8}$ of an inch, according to whether they are of copper, sheet-iron or cast-iron.

The number n of guide-vanes lies between 40 and 80 according to the diameter of the turbine, and if e denote the thickness of each, a section corresponding to ne on the development of the circumference must be subtracted from the section of the flow.

But this assumes that the water escapes in a radial direction; on the contrary, however, it flows obliquely, cutting the circumference at a certain inclination i; it is from these various considerations that the relation has been already established that:

$$Q = 2\pi r \times b \times V \times \frac{i}{\sqrt{1 \times i^2}} \times K \times m.$$

In this relation, K takes account of the reduction of section ne and m is the coefficient of contraction which provides for the actual reduction of the liquid jets.

Putting for simplicity:

$$2\pi \times \frac{i}{\sqrt{1 + i^2}} \times K \times m = A,$$

the above relation becomes:

$$Q = A \times r \times b \times V.$$

And as in the free deviation hypothesis:

$$V = \sqrt{2gH},$$

it follows that:

$$Q = A \times r \times b \sqrt{2gH}.$$

As the volume of flow Q of the stream to be utilised is known, as is also the height of the fall H, everything in the above relation is known save b which can therefore be immediately deduced thus:

$$b = \frac{Q}{A \times r \times \sqrt{2gH}}.$$

It should be noted in addition that the coefficient A which enters into the expression for the value of b depends on i, that is on the inclination of the speed V to the circumferential speed v of the wheel, or again on the obliquity of the last element of the guide-vane to the circumference of the guide-crown. *The value of this inclination is determined from the consideration of the speed desired* (§ 38).

There is direct connection between the speed and the efficiency, thus as has been seen, high-speed turbines are less efficient than low-speed ones. On the other hand, the former being necessarily smaller for the same power, have a lower first cost, and are superior from this standpoint.

However, having fixed the value of the inclination according to these various considerations, A can be calculated, and consequently the dimension b and all the other dimensions of the turbine may be determined.

Letting v represent the speed of the inner circumference of the moving crown, of radius r, the number of turns per minute is calculated from the general relation:

$$n = \frac{60v}{2\pi r}.$$

And since, as already established:

$$v = \mathrm{V} \times \frac{\sqrt{1 + i^2}}{2},$$

it follows that on replacing v by this value in the preceding formula:

$$n = \frac{60\mathrm{V}}{2\pi r} \times \frac{\sqrt{1 + i^2}}{2}.$$

The inner radius of the moving crown is equal to r, except for the clearance necessary between the two crowns. The depth of the blades at the entrance is theoretically equal to b; to correct for imperfections of workmanship and to prevent the water coming from the guide-channels from striking the cheeks of the moving crown, this dimension is always taken a little greater, and practically:

$$b_1 = b + \tfrac{1}{4} \text{ of an inch.}$$

The outer radius lies between $1{\cdot}2r$ and $1{\cdot}5r$ as for the fixed crown.

The depth of the moving crown b' at the outlet is calculated from the expression:

$$\mathrm{Q} = 2\pi r' \times b' \times m \times \frac{i_1}{\sqrt{1 + i_1^2}} \times \mathrm{K}' \times w',$$

whence:

$$b' = \frac{\mathrm{Q}}{\mathrm{A}' \times r' \times w'}.$$

On putting:

$$\mathrm{A}' = 2\pi \times \frac{i_1}{\sqrt{1 + i_1^2}} \times \mathrm{K}' \times m,$$

the coefficient K′ allows for the thickness of the blades, whose number varies from 30 to 60 but must never be greater than those of the guide-vanes, so as to allow foreign bodies entering the latter to pass freely.

The angle between the last portion of the vane and the tangent to the outer circumference should correspond to an inclination such that the condition is realised:

$$\frac{b'}{b} = \frac{r^2}{r'^2} \times \frac{i}{i_1} \times \frac{2\sqrt{1 \times i_1^2}}{(1 + i^2)}.$$

We know in fact, that this equation must be satisfied in order that no water may leak by the clearance between the two crowns, the turbine being assumed drowned.

But if the turbine is placed at a sufficient height above the tail-race level to run no risk of being drowned under ordinary circumstances, b' may be increased so that the ratio considered is greater than that given by the second member of this equation. Then the water will not entirely fill the channels of the moving crown, and if care is taken to arrange vents in the upper cheek of the crown, the turbine works as a free deviation or impulse turbine.

It is worth while considering the influence of the speed of rotation in the Fourneyron turbine on the volume of water passing through it.

It has already been established that:

$$w' = v \times \frac{r'}{r}.$$

Putting this value in the expression for the flow, we have:

$$Q = A' \times r' \times b' \times v \times \frac{r'}{r}.$$

This relation shows that the flow through the turbine is proportional to its speed of rotation, but for this to be so, the free deviation condition must be realised for all speeds; now the parts of the turbine cannot be modified to suit different speeds and consequently the proportionality which results from this relation does not exist in practice.

The flow through the turbine always increases to some extent however with increase of speed, whereas it would have been advantageous for it to have fallen off in order that the speed regulation might have been automatic.

41. Inward-flow Turbine. This turbine is installed in a chamber of water A as is the outward-flow turbine. In this case, however, there is no tube giving the water access to the centre of the turbine. The motion of the water is reversed, entering horizontally at the outer circumference of the fixed

crown, and converging radially towards the axis, in order to escape vertically at the centre of the moving crown (fig. 31).

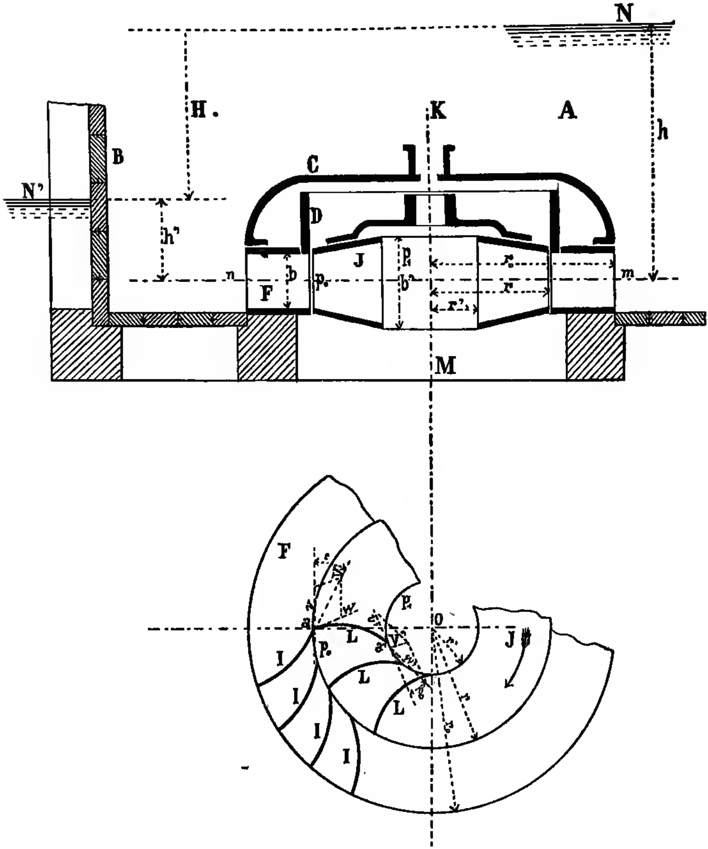

Fig. 31. Inward-flow turbine.

The guide-crown F is fixed at the bottom of the water chamber and the moving crown J is situated concentrically inside it. It is necessary therefore that the latter should be protected from direct access of the water to its centre; for

this reason the tube formerly used is replaced here by a sort of bell C.

This latter fixed by its edges to the outer perimeter of the guide-crown, is only pierced at its centre for the passage of the motor axle through a stuffing-box. The moving crown is suspended from a disc mounted on the axle.

In the clearance left free between the two crowns, a circular gate D can slide, which by being raised or lowered allows the flow to be regulated by throttling the passages to a greater or less extent; it also serves to stop the flow altogether when stopping the turbine.

As in the case of the outward-flow turbine, the fundamental equation is:

$$\frac{V^2}{2g} = \frac{p_a}{d} + h - \frac{p_0}{d},$$

where V is the absolute speed of the water at the point a, that is on entering the moving crown:

p_a is the atmospheric pressure:

p_0 is the hydrostatic pressure at the point a.

The triangle of velocities at the same point a gives as before:

$$w^2 = V^2 + v^2 - 2Vv\frac{1}{\sqrt{1+i^2}},$$

where i again denotes the inclination between the direction of V and the tangent v at the circumference of the moving crown.

If as before we use I to represent the inclination of V to w, and I_1 to represent that between w and v, we have just as in the outward-flow turbine:

$$\frac{v}{V} = \frac{I}{I_1} \times \frac{\sqrt{1 + I_1^2}}{\sqrt{1 + I^2}}.$$

Further, the triangle of velocities at a' gives the connection established in the previous case:

$$V'^2 = w'^2 + v'^2 - 2w'v' \times \frac{1}{\sqrt{1 + i_1^2}},$$

where i_1 again denotes the inclination between the relative speed w' and the circumferential velocity v' in the direction of the tangent at a'.

The efficiency may always be expressed by:

$$E = \frac{H - \frac{V'^2}{2g}}{H} = 1 - \frac{V'^2}{2gH}.$$

This equation shows that the efficiency is improved as V′ approaches zero; if indeed V′ were nothing the efficiency would be perfect and equal to unity.

But V′ cannot be equal to zero unless the moving vanes L are tangential to the circumference at a', and they cannot be so made without entirely suppressing the discharge of water into the interior of the moving crown J.

One must therefore be content to make the inclination i_1 as small as possible on the one hand, and on the other to satisfy the condition:

$$w' = v',$$

granted that the assumed speed v' is itself sufficiently great to provide for the total flow of the fall.

This relation allows the general equations to be simplified as before, and on this hypothesis, we get:

$$V = \frac{g \times H}{v} \times \sqrt{1 + i^2}.$$

Or in another form, giving the speed v of the outer circumference of the moving crown:

$$v = \frac{g \times H}{V} \times \sqrt{1 + i^2},$$

or again:

$$\frac{v}{V} = \frac{gH}{V^2} \times \sqrt{1 + i^2}.$$

The same condition holds for the inward-flow turbine working without reaction, as for the outward-flow turbine, namely:

$$\frac{p_0}{d} = \frac{p_a}{d} + h'.$$

If the crown were undrowned, that is to say placed above the tail-race surface N′, the value of h' would be zero and the stated condition would be reduced to:

$$\frac{p_0}{d} = \frac{p_a}{d}.$$

It is known that when the turbine works without reaction:

$$V = \sqrt{2gH},$$

and that:

$$w = v.$$

It follows that the ratio $\frac{v}{V}$ given above becomes simplified thus:

$$\frac{v}{V} = \frac{\sqrt{1+i^2}}{2}.$$

This ratio $\frac{v}{V}$ which depends only on the angle between the speeds and V, enables turbines to be classed as we have seen, according to their circumferential speed in terms of the absolute velocity of the water at the outlet of the guide-crown.

It is very important that the inward-flow turbine should be drowned. If in fact the mean plane of the moving crown be situated at a height h'' above the tail-race level, the actual fall will be reduced by a corresponding amount and will become $(H - h'')$.

Hence the efficiency will become:

$$E = \frac{H - h'' - \frac{V'^2}{2g}}{H} = \left(1 - \frac{V'^2}{2gH}\right) - \frac{h''}{H},$$

and the reduction in efficiency will be greater in proportion as h'' represents a greater fraction of H.

The useful work would be reduced in the same proportion. If Q denotes the number of cubic feet passing through the turbine per second, the expression for this work in the case of the drowned turbine is:

$$Tu = 62{\cdot}4 \times Q \times H \times E = 62{\cdot}4Q \times \left(H - \frac{V'^2}{2g}\right).$$

And in the case of a turbine raised above the tail-race level :

$$T'u = 62{\cdot}4Q \times \left(H - \frac{V'^2}{2g} - h''\right).$$

It is advisable therefore, with small falls, to place the moving crown below low-water mark in the tail-race. Thus the same conclusions are arrived at as for the outward-flow turbine.

The flow through the moving crown is always given by the relation :

$$Q = m \times K \times 2\pi r' \times b' \times \frac{i_1}{\sqrt{1 + i_1^2}} \times w'.$$

It therefore depends directly on the relative speed w' at the outlet of the moving crown. But w' itself depends on the outer peripheral speed of the moving crown. In fact it may be shown that :

$$w' = \sqrt{V^2 - v^2\left(\frac{i}{\sqrt{1 + i^2}} - \frac{r'^2}{r^2}\right)}.$$

Now the part in brackets being positive always, it follows that the term in v^2 is itself negative ; hence w' will diminish as v increases and *vice versâ*. Consequently the flow Q, which varies as w', varies inversely with the speed v.

This circumstance is greatly in the favour of the inward-flow turbine, for it makes it self-regulating. When the resisting torque falls off, the speed v tends to increase, but then the speed w' and with it the flow decrease, which reduces the work generated by the turbine, and consequently adjusts its speed to the standard value.

When, on the contrary, the resistance increases, v tends to decrease while w' and the flow through the turbine are increased, making the motive power grow in the required proportion to maintain the speed.

It will be remembered that this was not the case with the outward-flow turbine ; and this marked advantage in favour of the inward-flow turbine makes it much to be preferred

where the work to be done by the motor is liable to considerable changes and where the nature of the work requires a constant speed.

42. The Jonval suction tube. When the inward-flow turbine is drowned, the hydropneumatisation device described in connection with the outward-flow turbine, may be utilised for the purpose there mentioned.

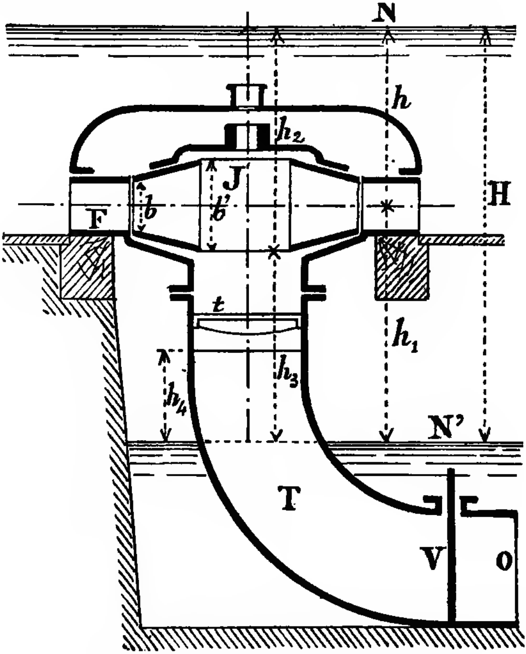

Fig. 32. Turbine with Jonval tube.

It was there seen that if h'' denoted the lowering of level from the surface in the tail-race to that of the water under the turbine dome, the fundamental equation became:

$$\frac{V^2}{2g} = H - (h'' - h'),$$

H being the fall or difference between the head-race and tail-race levels. This fall is reduced therefore by the quantity $(h'' - h')$, which is half the height of the moving crown.

It might, however, be found advisable to install the turbine above the tail-race level, but then the whole fraction of the fall corresponding would be lost. It is to remedy this disadvantage that Jonval devised the suction tube which is known by his name. Figure 32 illustrates this arrangement.

The guide-crown F is fixed to the bottom of the water chamber where the penstock canal ends. This crown is connected by a sort of conical funnel to a circular tube T which bends when it reaches the tail-race level and discharges horizontally at the bottom of this canal.

The vertical axle, on which the moving crown is mounted, turns in a footstep bearing supported by the cross-piece t which is in turn fixed to the sides of the tube T.

A gate V, placed near the outlet of the pipe, serves to regulate the flow and also to stop the turbine.

The water leaving the moving crown, falls in parallel stream-lines inside the Jonval tube T, in such a manner that the pressure at any horizontal section obeys the hydrostatic law. That is to say there are really two communicating vessels, constituted by the water chamber on the one hand, and the tail-race on the other hand, the communication being made by the down-pipe T.

Under these conditions, the hydrostatic pressure at the outlet of the moving crown at a' will be (§ 41):

$$p_1 = p_a - dh_3.$$

Now in order that there may be neither ingress nor egress of water through the clearance existing between the fixed and moving crowns, the condition must always be realised that:

$$\frac{p_0}{d} = \frac{p_a}{d} - h_1.$$

The fundamental equation:

$$\frac{V^2}{2g} = \frac{p_a}{d} + h - \frac{p_0}{d}$$

becomes in consequence :

$$\frac{V^2}{2g} = \frac{p_a}{d} + h - \frac{p_a}{d} + h_1 = H.$$

Obviously therefore the turbine uses the whole fall H, as if the moving crown were situated exactly at the tail-race level.

The essential condition which must be fulfilled in order that the Jonval tube may act properly, is that the surface in the tail-race must always be kept above the upper edge of the orifice O. When this condition is not satisfied, atmospheric pressure is no longer so transmitted as to sustain the column of water, and the latter breaks so that the water in the pipe tends to take up the level N of the tail-race. Thus the whole height h_3 is lost and the turbine uses nothing beyond the fall $\left(h + \frac{b'}{2}\right)$.

Further, the preceding relation :

$$p_1 = p_a - dh_3,$$

or :

$$\frac{p_1}{d} = \frac{p_a}{d} - h_3,$$

indicates that the column of liquid h_3 ought to be appreciably less than the height $\frac{p_a}{d}$ corresponding to atmospheric pressure; because if the pressure $\frac{p_1}{d}$ becomes too small, the air dissolved in the water is set free and has the effect of breaking the water column and reducing the useful fall.

This loss is always exceedingly small, if the break takes place just below the lower face of the moving crown. It is more important however in proportion as the break is made in a section nearer the tail-race surface.

In fact, in the first case, the equation always holds :

$$p_1 = p_a - dh_3,$$

which comes to the same thing as saying that the pressure p_1

at the outlet of the moving crown, increased by the pressure due to the column of water of height h_3, is in equilibrium with the atmospheric pressure at the tail-race level.

But if the column is broken at an intermediate section, at the height h_4, for example, above the tail-race surface, the hydrostatic pressure at the level N′ in the pipe is now only proportional to this height, and consequently:

$$p_1 = p_a - dh_4.$$

If finally the break occurred at the level N′ the relation would become:

$$p_1 = p_a - 0 = p_a,$$

that is to say the term to be subtracted would be zero and p_1 would become a maximum.

Now this term p_1 appears in the different relations giving the speeds which we have had to consider previously; for example the relative speed w' at the outlet of the moving crown is expressed by:

$$w'^2 = w^2 + v'^2 - v^2 + 2g\left(\frac{p_0}{d} - \frac{p_1}{d}\right).$$

This speed w' and consequently the quantity of flow will be correspondingly smaller as p_1 is greater.

Instead of considering the height h_4 of the section at which the break occurs, the distance $(h_1 - h_4)$ of this section from the median plane of the moving crown might be taken into account, putting:

$$h_1 - h_4 = z,$$

or:

$$h_4 = h_1 - z.$$

Now in general:

$$p_1 = p_a - dh_4 = p_a - d(h_1 - z).$$

Hence the expression for w' becomes:

$$w'^2 = w^2 + v'^2 - v^2 + 2g\left(\frac{p_0 - p_a}{d} + h_1 - z\right).$$

Now the fundamental equation:

$$\frac{V^2}{2g} = \frac{p_a}{d} + h - \frac{p_0}{d},$$

may be written thus:

$$V^2 = 2g\left(\frac{p_a - p_0}{d} + h\right).$$

And taking into account the expression for w', it becomes:

$$V^2 = w^2 - w'^2 + v'^2 - v^2 + 2g\,(H - z).$$

Similarly, in all the turbine equations, H must be replaced by $(H - z)$ and the turbine must be regarded as using only the fall $(H - z)$.

It follows that the theoretical efficiency is:

$$E = \frac{(H - z) - \frac{V'^2}{2g}}{H} = 1 - \frac{V'^2}{2gH} - \frac{z}{H}.$$

Obviously then the efficiency is smaller as z is greater, that is as the section where the break occurs is nearer to the tail-race surface level.

43. Calculation for an Inward-flow Turbine. The principal dimensions of an inward-flow turbine are estimated in the following manner.

It must be borne in mind that in such a turbine the water enters the moving crown horizontally, then the stream-lines curve and take a downward direction in order to flow past the section of the inner cylinder of the moving crown whose radius is r' (fig. 31).

Using u to represent the speed of the vertical stream-lines in this section, to obtain the radius r' of the cylinder, we have the relation:

$$Q = u \times \pi r'^2,$$

in which Q is the volume of flow per second which is known. Further u must be so chosen that it is considerably less than V, the velocity corresponding to the height of the fall.

For high falls generally:

$$u = \tfrac{1}{7}V = \tfrac{1}{7}\sqrt{2gH},$$

and for low falls we may take:

$$u = \tfrac{1}{5}V = \tfrac{1}{5}\sqrt{2gH}.$$

Knowing r', r and r_0 may be deduced from the connections:

$$r = 1{\cdot}5r' \text{ to } 2r',$$

and:

$$r_0 = 1{\cdot}15r \text{ to } 1{\cdot}25r,$$

the value of b the depth of the guide-vanes is given by the relation:

$$Q = m \times K \times 2\pi r \times b \times \frac{i}{\sqrt{1+i^2}} \times \sqrt{2gH},$$

and it is seen that the value b depends on the inclination i between the speed V with which the water enters the turbine and the peripheral speed v of the wheel; as this inclination ought to vary according to whether the turbine is low-speed or high-speed, the type desired will be fixed at the outset.

The inclination being thus determined, the speed v at the periphery of the moving crown may be deduced from it.

If n be the number of turns per minute, we must have:

$$n = \frac{60v}{2\pi r},$$

and moreover, since:

$$v = \frac{V}{2} \times \sqrt{1+i^2},$$

it follows that:

$$n = \frac{60V}{4\pi r} \times \sqrt{1+i^2},$$

in which formula the value of V is given by the relation:

$$V = \sqrt{2gH}.$$

The fixed blades are so shaped that the last inner portion is in the direction V and the curve thus traced cuts the outer circumference normally.

The height b' of the moving crown at the outlet is determined from the equation already established, which gives the volume of flow in terms of the relative velocity w' at this same point.

The essential condition that water may neither be lost

nor leak back through the clearance existing between the two crowns must be always satisfied, and consequently:

$$\frac{b'}{b} = \frac{r^2}{r'^2} \times \frac{i}{i_1} \times \frac{2\sqrt{1 + i_1^2}}{1 + i^2}.$$

In order to obtain the shape of the moving vanes, the parallelogram of velocities whose vectors v and V are known, is drawn at the points corresponding to the tips of the vanes, at a for example, so that the relative speed w is determined, the direction of which is the same as that of the first portion of the moving vane. The direction of the last portion of the vane nearest the inner diameter is given by w' which is obtained by drawing the parallelogram of velocities v', w', and V' at the point chosen.

The last portions of both the fixed and moving vanes are really made straight to ensure that the direction of the stream-lines at the entrance and the outlet of the moving blades shall be in the direction intended.

44. Turbines with Horizontal Axes. So far, only turbines with vertical axes have been spoken of. Both the outward-flow and the inward-flow turbine considered were arranged in this manner. They might equally well have been arranged on a horizontal axis and the inward-flow turbine is particularly well adapted for this arrangement.

The arrangement will be easily understood from fig. 33. The case A of the turbine is somewhat barrel-shaped and is flat near the axis. The guide-crown B is not placed concentric with the sides of the cover but is eccentric therewith, that is the centre of this crown which coincides with the axis of the turbine is nearer to the right side of the case by about a third of the difference in the diameters.

As a consequence of this arrangement the interval between the interior of the barrel and the outer rim of the guide-ring forms an annular spiral whose section constantly decreases in a very gradual manner from the entrance tube to the other extremity of the spiral.

It will be noticed that the water does not enter the guide-crown in just the same manner as in the case of the turbine with the vertical axle. In the latter, the crown being horizontal, all the channels opened at the same level, and water entered each channel under precisely similar conditions. This is no longer the case in the turbine in which the axis is horizontal and the moving crown vertical. The various channels

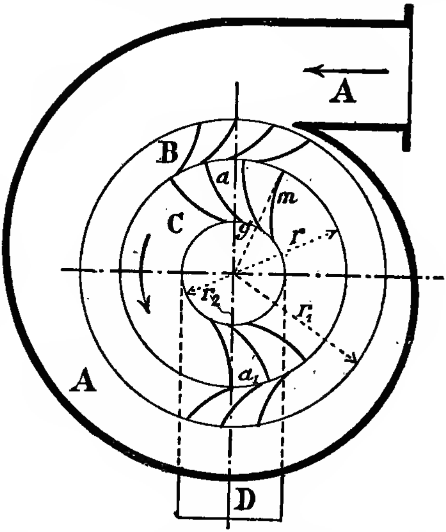

Fig. 33. Inward-flow turbine with horizontal axis.

are at different levels, and those which are at the upper extremity of the vertical diameter, and consequently nearest the admission pipe, receive the water first; then as the stream advances flowing downwards along the annular space the lower channels in the guide-crown are supplied.

After reaching the bottom of the barrel, the water of course rises again in the right-hand portion of the spiral space and feeds the corresponding half of the fixed crown. Thus the water flows in a closed circuit in the contrary direction to that of the motion of the hands of a watch.

Obviously, when the current after making a complete circuit again approaches the admission pipe, the sheet of water may be reduced to simply the amount necessary to supply the last channel. It is because of these considerations that the spiral form is given to the arrangement.

In fact, the radial section of the current in the annular space is proportional, at every point, to the volume of water which must be supplied to the development of the fixed crown beyond. The angle which the first element of the guide vanes makes with the outer circumference is so chosen that all the stream-lines preserve the velocity which they possessed in the supply pipe. This pipe coming from the head-race is bolted on to the admission pipe of the turbine barrel.

The water, after having passed through the inner moving crown, rushes radially towards the centre and escapes by a bent pipe D which is joined to the side of the barrel and whose diameter is equal to that of the inner circumference of the moving crown.

When the moving crown is rather large in breadth, it is advisable to strengthen it and to divide the body of water; to accomplish this the moving crown is made with a central cast-iron disc bearing blades on both faces so that the water on entering is divided between the symmetrical halves. At the outlet the water flows into the tail-race through two similar pipes D placed one on each side of the turbine barrel.

The connections between the various quantities which enter into the working of the turbine with the horizontal axis, are established from the same considerations as before, but account must be taken of certain modifications resulting from the particular arrangement of the crowns.

Assuming that the relation characteristic of working without reaction is realised, and that therefore:

$$\frac{p_0}{d} = \frac{p_a}{d} + h'$$

the relation which gives the value of the relative speed at the outlet of the moving blades is consequently:

$$w'^2 = w^2 + v'^2 - v^2.$$

Let us consider now, in the case of the turbine with the horizontal axis, two blades a_1 and a situated at opposite extremities of a diameter of the moving crown. In general the two blades will be at different levels, and moreover, while the water flows downward past the higher blade, it flows in the opposite direction past the lower blade.

Hence in the first case, the speed w' of the water will be increased by the acceleration due to the height of fall equal to the thickness of the blades, whereas in the second case this speed will be diminished by the same acceleration.

If this height of fall be represented by h'', then:

$$w'^2 = w^2 + v'^2 - v^2 \pm 2gh''.$$

It will be seen that h'' is only equal to the difference of the radii $(r_1 - r)$ for two blades situated at opposite ends of the vertical diameter. For other pairs of blades, the height h'' will be smaller as the corresponding diameter is nearer the horizontal diameter.

To obtain the general expression for this height of fall, let us take as an example the radius to the centre of the blade m say. This radius makes an angle g with the vertical diameter. The height through which the water will fall in this blade will be the projection of the blade m on the vertical, and if i be used to denote the inclination of this radius, we shall have for the blade under consideration:

$$h'' = (r_1 - r) \times \frac{1}{\sqrt{1 + i^2}}.$$

So that actually the previous relation ought to be written:

$$w'^2 = w^2 + v'^2 - v^2 \pm 2g(r_1 - r) \times \frac{1}{\sqrt{1 + i^2}}.$$

But it is easily seen that, when the turbine receives water over the whole of its circumference, the positive terms corresponding to the half of the moving crown situated above

the horizontal diameter are numerically equal, and pair with, the corresponding negative terms of similar values respectively, therefore these different terms cancel, and there is no need to take them into consideration as there would be if the turbine only received water over a portion of its circumference.

In order to avoid all loss of fall which would result from the lower opening of the pipes D discharging above the tail-race level, which loss of fall would be equal to the vertical distance between the tail-race level and the centre of the turbine, the Jonval tube device which has been already described, is made use of.

In reality the barrel ought not to have a circular but an Archimedean spiral form. This would be the correct form for the section of the annular passage to be proportional to the flow through it, and the speed of the water would then be constant at all sections of the barrel.

To draw this shape, the radius corresponding to each diameter of the barrel must be determined.

If we use Q to denote the volume of flow in any section whatever, l to represent the breadth of the barrel, r_1 to represent the outer radius of the crown, and R to represent the radius vector of the barrel in the section under consideration, then:

$$Q = (R - r_1) \times l \times v.$$

As one passes from one meridian section to the next, the increase q of the volume of flow ought to be proportional to the increase of orifice surface of the guide-crown, which in turn depends on the increment of the angle a at the centre which the radius vector makes with the origin; hence using K to represent a constant coefficient, then:

$$q = K \times a.$$

But in addition if r represent the corresponding increase of the radius vector:

$$q = r \times l \times v,$$

whence:

$$K \times a = r \times l \times v,$$

and:

$$r = \frac{K}{l \times v} \times a.$$

Now at the origin the radius vector reduces to r_0, in general therefore after it has been increased by r it becomes:

$$R = r_0 + \frac{K}{l \times v} \times a.$$

The coefficient K still requires to be determined. The total volume of flow Q is of course built up of the sum of the successive increments q, so that as the angle described by the radius vector varies from 0 to 2π, we have:

$$Q = K \times 2\pi.$$

And since this volume must pass through the openings in the guide-crown, we have also, after deducting the space occupied by the N vanes of thickness e:

$$Q = (2\pi r_0 - Ne) \times l \times v = K \times 2\pi,$$

whence we may deduce:

$$K = \frac{(2\pi r_0 - Ne) \times l \times v}{2\pi}.$$

And on replacing K in the expression for R, it becomes finally:

$$R = r_0 + \frac{2\pi r_0 - Ne}{2\pi} \times a,$$

or neglecting Ne:

$$R = r_0 + r_0 \times a = (1 + a) r_0.$$

As stated above the case is so calculated that all the stream-lines cut the outer circumference of the crown at the same angle. To attain this condition, it is necessary that the portion of the case immediately to the right of the radius vector passing through the extremity of a fixed blade should make a constant angle with the normal to this radius vector. If now this same angle be adopted for the direction of the first portion of the fixed blade, the stream-line guided by the case will cut the outer circumference in the manner stated above.

To construct the development, a circle is drawn concentric

with the turbine and whose radius is determined by the equation:

$$r = r_1 \times \frac{i}{\sqrt{1 + i^2}},$$

where i denotes the inclination between the speed V and the tangent to the circumference of the moving crown. The guide-blades themselves are formed of fractions of the same development.

It is rather difficult to construct the case as a development of a circle in cast-iron, and hence this shape is usually replaced by a circular barrel placed eccentric with the guide and moving crowns. The result is rather less favourable, but the simplicity and economy in construction are very much greater.

It is advisable to so calculate the section of admission that the speed of the water never exceeds 10 or 12 feet per second, in order that losses of head due to friction against the sides may be reduced as much as possible.

45. Parallel-flow turbine. In the types previously studied, water was directed radially through the motor either from the centre to the periphery, or *vice versâ*. The parallel-flow or axial turbine now under consideration is so-called because in both the guide-blades and in the moving crown the water flows parallel to the axle of the motor.

These turbines generally have vertical axes. In this case, the two crowns are superposed, as shown in fig. 34.

The turbine is placed in a water-chamber whose sides are continuations of the sides of the supply canal. This chamber is closed on the down-stream side by a partition B. The frame-work constituting the bottom of the said chamber is supported on the side-walls of the canal, on the up-stream sustaining wall, and on columns which rest on the bed of the tail-race.

The guide-crown is bolted to the bed of the water-chamber. The moving crown J, which is placed directly below, is furnished with a nave mounted on a hollow cast-iron axle G,

which in turn is hung from a pivot above, which turns on the tip of a central wrought-iron spindle supported from the bed of the tail-race.

The sheath P has for its object the prevention of water flowing from the head-race through the bushing in the fixed crown which allows the hollow axle to pass through, and the latter projects above the floor to transmit the power of the turbine.

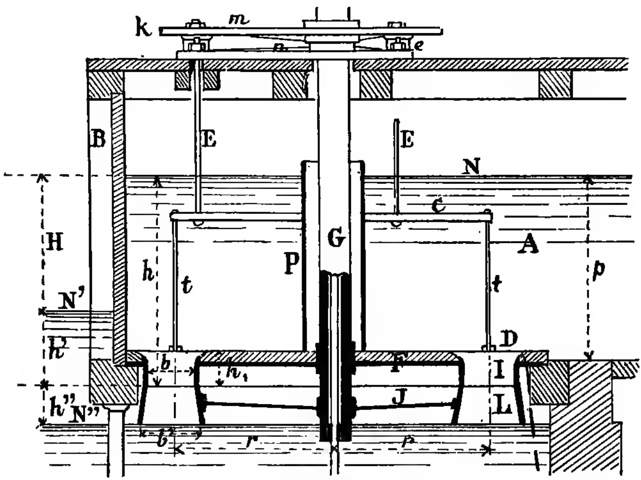

Fig. 34. Parallel-flow turbine.

This turbine is provided with a gate system consisting of vertical diaphragms placed behind each guide-blade. All these gates are suspended by rods *t* from a ring *c* which is itself supported by three bars E screw-threaded at their upper ends. Each of these bars is furnished with a nut *e*, surmounted by a toothed pinion *k*.

The three pinions gear with a wheel *m* which drives them all simultaneously. If the wheel be moved in one sense or the other the nuts *e* are turned making the screws and consequently the gates D rise or fall.

In this way by opening the gates more or less, either the

turbine may be stopped or the amount of water flowing through it may be regulated.

The connection between the various speeds to be considered in the working of the turbine, are like those relating to the inward-flow turbine; the sole difference lies in the fact that the radii of the two crowns being the same, it becomes necessary to put $r' = r$ in the previous formula.

The particular equation giving the speed V with which the water flows away from the moving crown will be:

$$V^2 = w^2 - w'^2 + 2gH,$$

where w is the relative speed with which water enters the moving crown, w' is the relative speed of the water at the outlet of the crown and H is the difference of height between the head-race level N and the tail-race level N′.

Assuming that $w' = v$ it will be found that:

$$v = \frac{H}{V} \times g \times \sqrt{1 + i^2},$$

where i represents the inclination of the relative speed w to the circumferential velocity v of the turbine.

In order that there may be neither loss of water nor any water flowing back into the moving crown from the tail-race through the clearance which must be left between the two crowns to avoid undesirable friction, the condition that must always hold is:

$$\frac{p_0}{d} = \frac{p_a}{d} + h',$$

that is to say, the hydrostatic pressure exerted from above downwards on the plane separating the two crowns, must be in equilibrium with the pressure in the opposite direction due to atmospheric pressure plus that due to the depth of this plane of separation below the tail-race surface.

It may be shown that in order for this to be the case, it is necessary that the above-mentioned angle of inclination i must be at least equal to half the angle of inclination of the last portion of the moving vane to the horizontal.

If the above condition be fulfilled, the fundamental relation:

$$\frac{V^2}{2g} = \frac{p_a}{d} + h - \frac{p_0}{d},$$

becomes:

$$V^2 = 2gH,$$

or:

$$V = \sqrt{2gH},$$

and the relation given further back:

$$v = \frac{H}{V} \times g \times \sqrt{1 + i^2},$$

reduces to:

$$v = \frac{V \times \sqrt{1 + i^2}}{2}.$$

These are the conditions corresponding to the case of a turbine working without reaction.

The expression for the efficiency is simplified when the equality of the radii r and r' is taken into account, and we get:

$$E = 1 + \sqrt{\frac{1 + i^2}{1 + i_1^{\,2}}} - \sqrt{1 + i^2},$$

where i is the inclination of w to the horizontal speed v, and i_1 is the inclination of the latter to w'.

Suppose, for example, that the two inclinations i and i_1 are equal to 0·58 which corresponds to an angle of 30 degrees, the efficiency then would be:

$$E = 1 + 1 - \sqrt{1 + 0{\cdot}33} = 2 - \sqrt{1{\cdot}33} = 0{\cdot}85,$$

that is, the efficiency would be 85 per cent.

As a matter of fact the efficiency of these turbines varies from 93 to 73 per cent. according to the class of turbine, low-speed or high-speed.

As the volume of the stream falls off, the flow through the turbine must be reduced by partially closing all the gates, so that the level N may be kept at a constant height. But as a result a contraction of the jets occurs under the gates, followed by a sudden expansion at the entrance to the moving crown; the effect is to so reduce the effective head that the efficiency

may be diminished from 15 to 20 per cent. Hence this system of gates is not suitable for streams whose flow is variable.

It has been assumed up to this point that the level N′ of the tail-race was higher than the plane separating the two crowns. But it might be at the same height or even lower than the lower face of the moving crown.

On referring to the figure, it will be seen that in the latter case the fall becomes:

$$H = h + h''.$$

The relation characteristic of non-reaction working is therefore simply:

$$\frac{p_0}{d} = \frac{p_a}{d};$$

and it also follows that:

$$V = \sqrt{2gh},$$

which shows that, as must be the case in such a turbine, the absolute velocity with which the water in the turbine is flowing is that due to the total height of fall h.

It should be noticed that h never actually constitutes the complete height of the fall, since the moving crown is situated above the tail-race level by the whole of the height h''. But it must be noted that in the parallel-flow turbine this fall is not lost, since it is utilised in accelerating the water relative to the vanes of the moving crown.

It must not be concluded from this that the height of the moving crown may be given any value whatever, for the efficiency depends on this height, and it is advantageous to make h'' very small in relation to the total fall H, when the turbine works undrowned.

When the parallel-flow turbine is installed at a point in a fall intermediate between the head-race and tail-race levels, the whole fall may be utilised nevertheless, if use be made of the Jonval device described in connection with the inward-flow turbine.

If the turbine is to work drowned, the advantages of free

deviation may likewise be procured by adopting the Girard hydropneumatic device, which lowers the water-level so as to clear the moving crown, by means of compressed air.

Always in such a case we have:

$$V = \sqrt{2g(H - h'')} = \sqrt{2gh},$$

a result which is analogous to that obtaining when the moving crown is situated above the tail-race level.

46. Regulating Devices. The system of small vertical gates used in the Fontaine or parallel-flow turbine (fig. 34) is known under the name *part gate,* because the baffles only partially close the passage section in the guides. As stated above this device produces contraction of the water jets where they pass under the partially closed gates, followed by sudden expansion in the channels of the moving crown, which gives rise to loss of head and consequently diminishes the efficiency of the turbine. This reduction may be as much as 15 or 20 per cent. when the flow is halved.

Because this system of regulation is not suitable for a turbine placed in a variable stream, when the flow through the motor must be reduced to maintain the level N at a constant height, several contrivances have been designed to get over the disadvantage.

For instance, Koechlin attempted to restrict the breadth of the orifices throughout the whole height of the buckets by inserting obstructing wedges in them so as to reduce the flow. But this method did not allow the flow through the turbine to be varied in a continuous manner as the stream varied and it had the disadvantage of requiring the turbine to be stopped to put the wedges in place or remove them.

With the same object Fontaine constructed turbines with double concentric crowns cast together. The fixed crowns were controlled by two independent systems of gates which could be manipulated separately or together according to the flow.

The slide-gates devised by Girard consisted of a certain

number of flat baffles, in the form of segments, arranged horizontally over the whole area of the guide-crown. By means

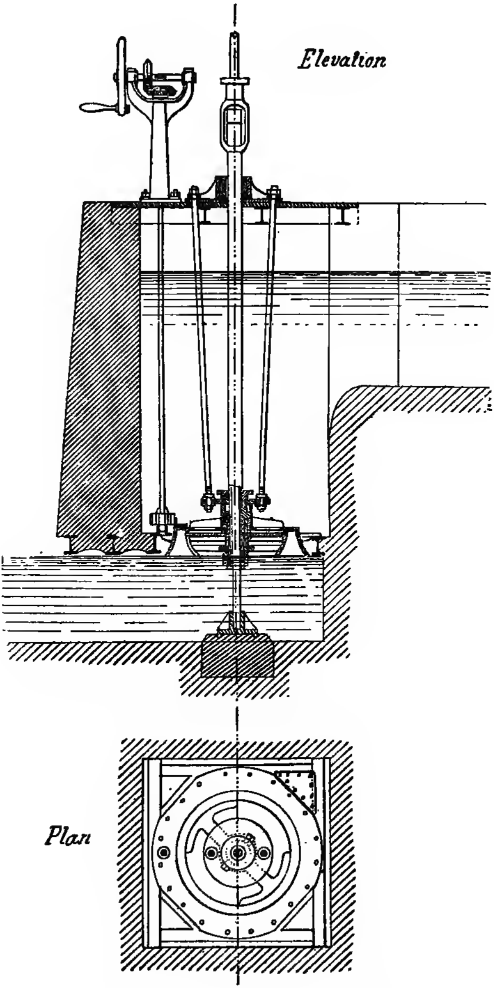

Fig. 35. Turbine with wing-gates.

of a suitable mechanism the slides diametrically opposite each other were made to move two at a time along the radii of the turbine, so that the corresponding orifices were completely opened or closed. In this way the contraction due to partial reduction of the orifices was avoided, but the flow through the turbine could only be changed by an eighth, a tenth or a twelfth, according to the number of vanes, and not in a continuous manner.

Another contrivance due to Girard is the *double butterfly* gate. It is composed of two circular segments each covering a quarter of the guide-crown and connected by radial arms supported from a hub which can turn round on the boss of the guide-crown (fig. 35). The motion is controlled by a hand-wheel which actuates a pinion shaft, the pinion gearing with a toothed wheel attached to the segments.

The guide-crown only possesses openings in two opposite quadrants, the two others being closed and receiving the butterfly wings when the gate is fully opened.

In the case of a stream with a very variable flow, it is preferable, from the point of view of efficiency, to open the small number of orifices corresponding to extreme low water on one side only of the turbine. This is done by making the two wings independent and controlling each by a special gear train.

Fig. 35 shows the general arrangement of a three horse-power parallel-flow turbine provided with a double wing-gate.

This system has the disadvantage of doubling the size of the turbine for a particular power since it does not permit the turbine to be fed over more than half the circumference.

The Fontaine *scroll gates* enable the whole of the turbine to be in use at once. A toothed wheel capable of turning round on the boss of the guide-crown is provided with two extended arms which end in sockets, in which are axles situated on the same diameter carrying two rollers in the form of truncated cones (fig. 36). Two circular rubber bands or scrolls are fixed by one end each to opposite extremities respectively of the same diameter of the guide-crown, and the

other ends of the two bands roll on the two rollers, one on each.

On turning the toothed wheel by means of the controlling mechanism the two scrolls are rolled up on their respective

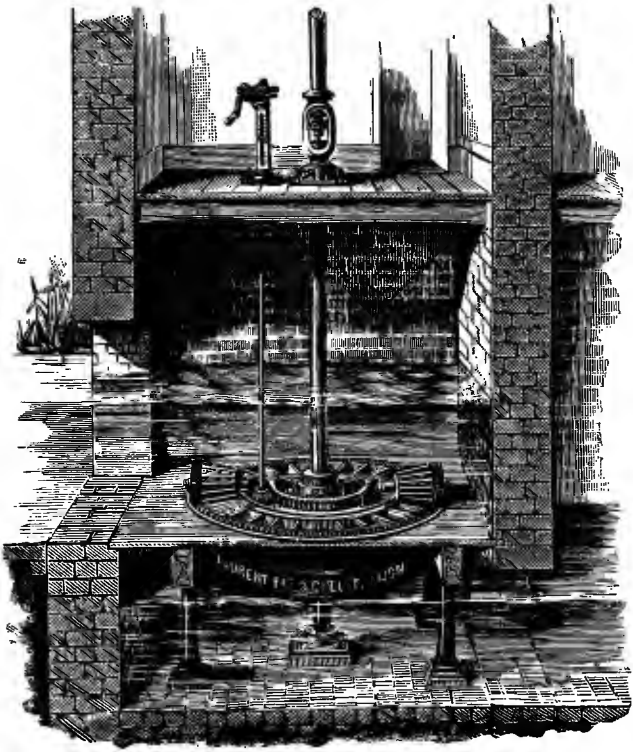

Fig. 36. Scroll gates.

rollers and open all the orifices in the guide-crown in succession; in this way the turbine may be fed over the whole of its area, save for the space occupied by the rollers at the end of their travel.

Mention ought also to be made of the system in which the

small vertical gates are replaced by clack-valves capable of opening or closing by rocking on horizontal hinges; finally there are the American gates constituted by the guide-vanes themselves being pivoted between the two annular plates of the fixed crown and controlled by radial motion rods.

47. Different arrangements of parallel-flow turbine. *Turbine with siphon.* For low falls it may happen that the depth of water over the guide-crown is less than three feet. In such a case eddies and whirls will be formed above the orifices, with the result that both the flow and the efficiency are reduced.

As a consequence the turbine must be submerged, and then it must be undrowned artificially by compressed air, in order to ensure its working in time of drought, when in consequence of the diminution of flow it can only be supplied with water over a part of its circumference. Further the submerging of the machine entails excavation of the tail-race, which is often a difficult and costly operation.

To overcome these defects, Girard suggested supplying the fixed crown through a siphon whose up-stream portion rose above the upper level and was then joined to an annular piece which in turn joined on to the guide-crown. The depth of water covering the openings was thus increased by the difference between the upper stream level and the top of the siphon. The latter was primed by the aid of a small air-pump whose suction tube was connected to a tube cast on the highest point of the siphon.

Partial admission. The dimensions of a turbine for a given power, are smaller in proportion as the fall is higher and the volume of flow is less. In the case of very high falls therefore it would be necessary to use turbines of very small diameter, having very restricted openings which would be liable to be choked, and requiring in consequence of their speed very carefully designed and well-adjusted bearings.

This difficulty has been overcome by using turbines which are only supplied with water over a part of their circum-

ference. Fig. 37 shows the device designed by Girard for this purpose.

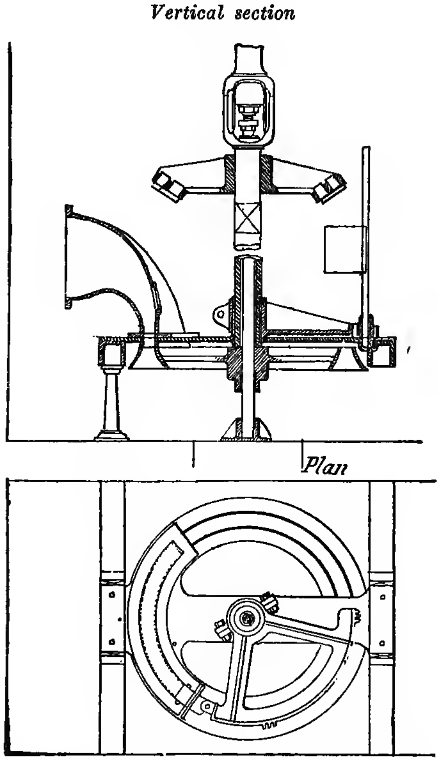

Fig. 37. Turbine with partial admission.

The horizontal supply pipe is bolted to a bent nozzle or injector whose outlet is developed in the form of a segment of the circular crown, and covering about a third part of this crown.

The gate is composed of a circular slide connected to a toothed sector which is pivoted on the boss of the fixed crown and whose teeth are in gear with a pinion, mounted on a vertical shaft and controlled by a hand-wheel.

This turbine must never be drowned and therefore must be installed above high-water mark in the tail-race.

The equations referring to the undrowned parallel-flow turbine apply equally well to the partial injection turbine, with the reservation that in the expression for the total flow Q, it must be borne in mind that only a definite fraction of the circumference is supplied.

If two falls of different height are available, the turbine may be supplied by two side injectors one for each fall, feeding two symmetrical segments of the fixed crown. Care must be taken under such circumstances that the guide-blades are so shaped as to obtain equal values for the speed of rotation v and for the relative speed w for the two sections of the turbine.

Turbine with horizontal axis. Like the other types the parallel-flow turbine may be mounted on a horizontal axis. The machine is in this case entirely enclosed in an envelope forming part of a siphon, as in the Jonval turbine arrangement (fig. 38).

The siphon is composed of three parts: the supply pipe on the right, the barrel containing the turbine, and the vertical pipe of the Jonval device which dips into the tail-race.

The horizontal axle of the turbine turns in three bearings, of which one is fixed to the inside of the case.

One of the two outside bearings serves to take the horizontal thrust of the water on the moving crown. The transmission pulley is mounted between the latter two bearings.

With these turbines, neither the part-gate device nor free deviation which implies rotation in air, can be used. Hence this system cannot be used advantageously for a stream of variable flow.

In the case of a high fall the partial admission device may be used with this turbine; it is advisable in this case in order to avoid all loss of fall to immerse the moving crown by an amount equal to the arc of injection.

48. Mixed turbines. In the types of turbine examined so far, the water either flows radially in the turbine, flowing

either from or towards the axis (outward-flow and inward-flow turbines), or flows parallel to the axis (the parallel-flow tur-

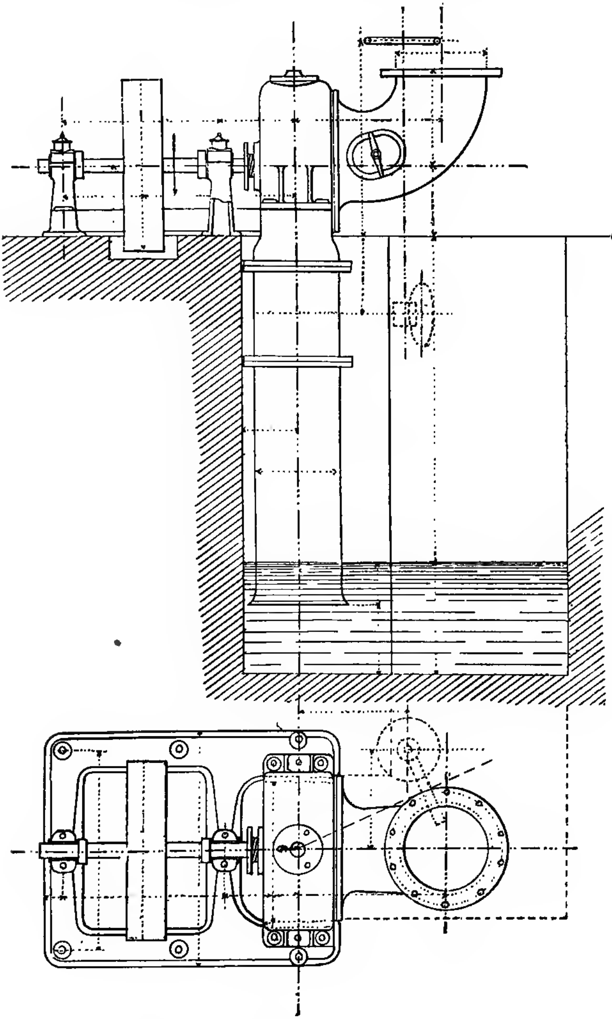

Fig. 38. Turbine with horizontal axis.

bine). Some years ago turbine constructors, more especially in America, brought out a system which combines in one turbine, both the inward-flow and the parallel-flow types, and to this has been given the name of *mixed turbine.*

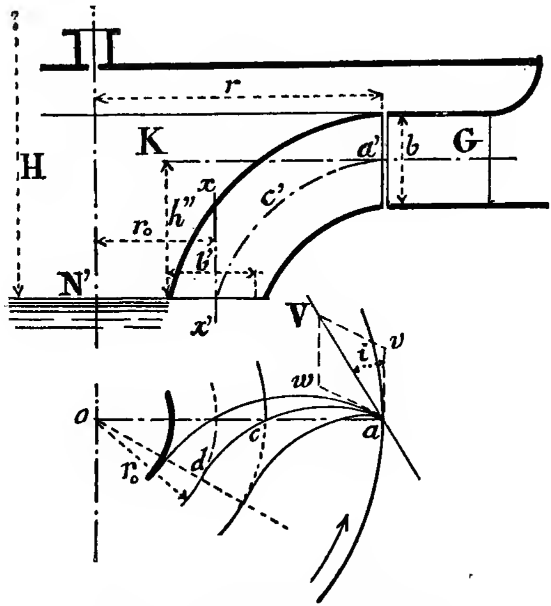

Fig. 39. Principle of the mixed turbine.

Fig. 39 shows the theoretical arrangement of such a turbine. The outer fixed or guide-crown G is arranged as usual, but the construction of the moving vanes is altogether different. These vanes in fact possess double curvature and their whole form is somewhat *helicoidal* or *spoon-shaped.*

The water coming from the guides first enters the moving crown radially flowing towards the axis, then it falls in a direction more and more parallel to the axis as shown by the line $a'c'x'$ in vertical projection and acd in horizontal projection.

At a and a' the first element of the vane is tangential to the direction of the relative velocity w. It will be observed moreover that the mean line of the vane, curving in the direction acd, is such that the last element of the vane at d always makes a very small angle with the horizontal plane.

The absolute velocity V with which the water enters the moving crown makes an angle whose inclination is i with the speed v of the outer circumference of the crown.

The equations which apply to the working of this turbine are easily deduced from those relating to the inward-flow turbine.

When the water is subject to no reaction, the condition of equality between the hydrostatic pressure at a and the atmospheric pressure is satisfied, and:

$$\frac{p_0}{d} = \frac{p_a}{d},$$

and from this it follows that the relative speed w and the speed v are equal also.

Now $$V^2 = 2g(H - h''),$$

since the absolute velocity at the outlet of the guide-blades is due to depth of $(H - h'')$ at the level GK.

The relative velocity with which the water enters the moving crown is given by the relation:

$$w^2 = V^2 + v^2 - 2 \times V \times v \times \frac{1}{\sqrt{1 + i^2}},$$

i being the inclination between the peripheral speed v and the absolute velocity V as stated above.

The relative speed of the water at the outlet of the moving crown becomes:

$$w'^2 = w^2 + v'^2 - v^2 + 2gh''.$$

It will be noticed that compared with the case of the inward-flow turbine, the speed w' is increased by the velocity due to the height of fall h'' between the levels GK and N'.

The absolute velocity of the water at the outlet of the moving crown is always such as to satisfy the expression:

$$V'^2 = w'^2 + v'^2 - 2w'v' \times \frac{1}{\sqrt{1 + i'^2}},$$

where i' represents the inclination of the last element of the moving vanes to the horizontal plane.

Finally the theoretical efficiency of the turbine will again be:

$$E = 1 - \frac{V'^2}{2gH}.$$

The mixed turbine may be placed with its axis either horizontal or vertical, and may be installed with the Jonval device at a point in the fall intermediate between the head-race and tail-race levels. It is also well adapted for use with the hydropneumatic device which enables it to work even when the tail-race is in flood.

In mixed turbines, the guides of the fixed crown are straight and their first portions instead of being normal to the outer circumference are inclined to it. However the guides are not simply plane surfaces. To diminish the contraction on entering the fixed crown, they are in fact given a lenticular form, or at any rate they are made taper towards the interior.

The first dimension to be determined is the diameter of the turbine's discharge pipe. For this purpose the loss due to the outlet velocity V′ is fixed; suppose this loss is 4 per cent. of the theoretical work, then:

$$E = 1 - \frac{V'^2}{2gH} = 1 - 0{\cdot}04,$$

whence:

$$V'^2 = 0{\cdot}04 \times 2gH,$$

and:

$$V' = 0{\cdot}2\sqrt{2gH} = 1{\cdot}6\sqrt{H}.$$

The section of the discharge pipe will be therefore:

$$S = \pi r^2 = \frac{Q}{V'} = \frac{Q}{0{\cdot}2\sqrt{2gH}},$$

whence:

$$r = \sqrt{\frac{Q}{0{\cdot}2\pi\sqrt{2gH}}}.$$

If the turbine works without reaction, it follows from the condition $v = w$, the triangle of velocities being isosceles, that the speed V is in the direction of the bisectrix of the angle formed by the speeds v and w, and if g be the angle between

w and the tangent at a, that is between w and the speed v, and using i to represent the angle between v and V, we have:

$$g + 2i = 180 \text{ degrees},$$

whence:

$$g = 180 - 2i.$$

As i generally varies between 25 and 30 degrees, obviously g varies itself in this case from 130 to 120 degrees. Again therefore we have what is characteristic of the impulse turbine, for this condition inserted in the general equations, produces the relation:

$$V = \sqrt{2gH},$$

which gives the value of the absolute speed of the water at the outlet of the distributor in the case of non-reaction working.

Let us use n to denote the number of guides, e their thickness, i the inclination of the current on entering the moving crown, and m to represent a coefficient of contraction which varies from 0·80 to 0·95 according to the nature and condition of the guide surfaces. The horizontal dimension of the outlet section of the distributor is simply the projection of the arcs of the circumference on the normal to the absolute velocity V; the total breadth is therefore:

$$l = 2\pi r \times \frac{i}{\sqrt{1 + i^2}}.$$

But the thickness of the blades must be subtracted so that the total section of the current will be:

$$S = \left(2\pi r \times \frac{i}{\sqrt{1 + i^2}} - ne\right) \times b.$$

And the volume of flow will be given by the relation:

$$Q = m \times 2\pi r \times \left(\frac{i}{\sqrt{1 + i^2}} - ne\right) \times b \times V.$$

In this expression, all the quantities are known save b, which can therefore be determined.

In a reaction turbine the foregoing relations are not satis-

fied; the angle g is therefore less than 120 degrees and the velocity V is less than would be due to the height of fall H.

It will be remembered that the name degree of reaction was given to the expression:

$$K = \frac{V}{\sqrt{2gH}},$$

whence:

$$V = K \times \sqrt{2gH}.$$

Let us assume that g is 90 degrees say, then the triangle of velocities is right-angled and we have:

$$v = V \times \frac{1}{\sqrt{1 + i^2}}.$$

Under these circumstances it will be found that the degree of reaction is of the value:

$$K = \frac{0{\cdot}707}{\sqrt{1 + i^2}}.$$

Since the actual speed varies between 0·8 and 0·9 of the theoretical speed, we may finally write:

$$V = 0{\cdot}8 \text{ to } 0{\cdot}9\ K \times \sqrt{2gH}.$$

The angle whose inclination is i, lying between 25 and 30 degrees, K will therefore vary from 0·750 to 0·815 and the velocity V will be on the average:

$$V = 0{\cdot}650\ \sqrt{2gH}.$$

It follows that, in the case under consideration, the section of the passage through the distributor must satisfy the equation:

$$Q = \left(2\pi r \times \frac{i}{\sqrt{1 + i^2}} - ne\right) \times b \times 0{\cdot}650\ \sqrt{2gH}.$$

In the same manner the height b of the guide-crown will be determined. The circumference of radius r will then be divided into as many parts as there are to be guide-blades and the latter will be made with an inclination of from 20 to 30 degrees with the inner circumference.

The shape of the moving vanes still remains to be drawn; this process, which is necessarily very complicated, is described in principle in the next paragraph.

49. Shape of moving blades. In ordinary turbines, the moving vanes are formed of cylindrical or conical surfaces, and the outlet edge is always straight and equal in breadth to the moving crown. We have seen that in mixed turbines on the contrary, the vanes take a helical or spoon-shaped form, so that the outlet perimeter is curved and has a very large development.

Mixed turbines are further generally characterised by the fact that the relative speed of the water entering the moving crown is in a direction normal to the circumference that is to say radial, while in other turbines the relative speed w is more or less inclined to it.

The angle g is therefore 90 degrees, while in the Girard turbines it lies between 120 degrees and 130 degrees.

Fig. 41 represents the horizontal projection of a moving blade of a mixed turbine. It will be seen that the first portion of the inner surface of the vane is normal to the tangent v, since it must be itself tangential to the relative velocity w.

The first portion of the outer surface of the vane is tangential to the line $a'm$, which makes an angle of 10 degrees with the direction of the speed w.

The shape of the outlet perimeter of the vane, and consequently the shape of the whole vane, is rather difficult to obtain.

First of all, one is given the vertical profile of the vane, like that shown in fig. 40, in which the contour EDBAA′B′C′E′ represents the projection of the vane on a vertical plane passing through the axis of the turbine.

To obtain the horizontal projection, operations are carried out as follows. The breadth AA′ of the vane is divided into a certain number of equal parts; to each of these parts there are corresponding sheets of water, whose mean axes are

situated at various distances r_1, r_2, r_3, r_4 from the axis of rotation XY.

Let us consider more particularly the stream-line whose projection follows the vertical line number one; its relative speed on entering the moving crown is w, and it leaves the blade at the point where this vertical line meets the line AA′; at this point its relative velocity is w_1'. Now this speed depends on the relative speed on entering w, on the pressure due to reaction, and on the centrifugal force.

If for V, the absolute velocity on entering, we take the average value:

$$V = 0{\cdot}650 \sqrt{2gH};$$

and for the angle between V and v, the value 20 degrees, we have as the relative speed w_1' at the outlet of the particular section under consideration:

$$w_1' = \sqrt{27H + v_1^2}.$$

For the stream-line at distance r_2 from the axis of rotation, we have similarly:

$$w_2' = \sqrt{27H + v_2^2},$$

and so on.

Now v_1, v_2, etc. are the circumferential velocities of concentric cylinders in the turbine, whose radii are respectively r_1, r_2, etc. These velocities are known because they are proportional to the radii. But the relative outlet velocities w_1', w_2', etc. are the hypotenuses of velocity triangles, whose other two sides are the constant value $\sqrt{27H}$, and the various speeds v_1, v_2, etc. at the points under consideration. Hence it is easy to calculate the outlet speeds, or to draw geometrically the lengths representing these speeds to a definite scale.

Knowing these speeds, it is necessary to compound them at each of the points 1, 2, 3, 4, 5, etc., with the absolute velocity V, and the corresponding circumferential velocities v_1, v_2, v_3, v_4, etc.

The velocity V′ ought to be the same for all the stream-

lines at the outlet of the turbine; it depends on the efficiency and this has been fixed already at the value (§ 48):

$$V = 1{\cdot}6 \sqrt{H}.$$

For example, for the stream-line at distance r_1 from the axis, the triangle of velocities will be drawn in the vertical plane at the point where the vertical number 1 meets the line AA′, which is one of the points on the vertical projection of the lower edge of the vane. This triangle will be formed by the speed v_1 possessed by the parts of the turbine situated at the distance r_1 from the axis of rotation, by the relative velocity w_1 of the corresponding stream-line, and by the common absolute velocity V′.

On constructing this triangle, of which the three sides are known, the inclination of w_1 to v will be obtained, that is to say, the direction of the last element of the vane, or its inclination to the horizontal will be known. Similar operations should then be carried out for the various stream-lines situated at distances r_2, r_3, r_4, etc. from the axis.

From these results the cylinders of radii r_1, r_2, r_3, etc. will be developed, and observing for each outlet point the angle made by the last element of the blade with the horizontal tangent at the same point, the limiting curve of the horizontal projection of the vane will be determined.

It will be necessary afterwards to examine whether the outlet section left free between two consecutive vanes allows sufficient room for the volume Q which must be discharged from the turbine.

In mixed turbines, the ratio $\frac{b}{r}$ of the height of the guide-crown to the radius of the moving crown is always very great; in fact, instead of being equal to 0·1 as a maximum, as is the case with Girard turbines, it is often nearly equal to unity. As a result, the mixed turbine is very much smaller as regards breadth; it is therefore more compact in form and more economical in construction.

It has also been shown that the outlet perimeter is greatly

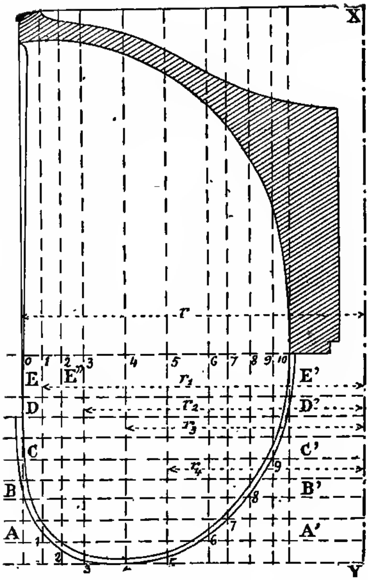

Fig. 40. Diagram of blade shape, vertical projection.

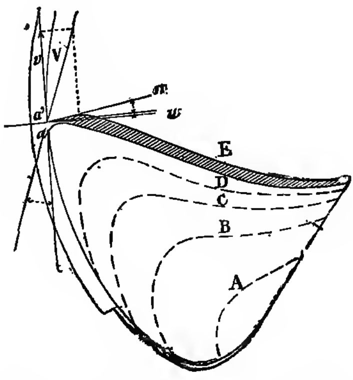

Fig. 41. Diagram of blade shape, plan.

developed, since it follows a curved contour like a spoon or screw blade; consequently the outlet section is considerable. This circumstance allows the outlet angle of the water to be reduced to a very great extent, so that the lower edge of the blades may be very acutely inclined with the horizontal.

This spoon-like curvature of the vanes is of great advantage, when the turbine is drowned, for it presents very little resistance to the motion of the turbine through the water.

In ordinary turbines care must always be taken to leave as little clearance as possible between the fixed and moving crowns. This is not necessary with mixed turbines which always have considerable clearance. This clearance may exceed one inch and is used for the passage of the cylindrical gate used to regulate the flow.

As a result of this arrangement the sheets of water are not guided by the fixed blades right up to the entrance to the moving blades, and it might be supposed that this unusual clearance would interfere with the working of the turbine. But it appears on the contrary that the ring of water thus obtained between the two crowns forms a medium in which the jets escaping from the guide-channels are collected and in which eddies caused by the thickness of the blades are reduced and lost.

50. Pivots and gates. Mixed turbines which have as their chief advantage the simplicity of their construction, are generally provided with axles supported from drowned bearings situated in the tail-race. This type of axle is in fact of a much cheaper construction than the ordinary hollow axle turning on a steel pin in the air.

The drowned bearing consists simply of a bushing of lignum vitae or of oak, fixed in a footstep, in which the end of the axle turns. Such a bearing does not require any oiling, lubrication being assured by the tail-race water.

But this system possesses the serious inconvenience that it is not easy to get at the bearing to examine it. The wooden

bearing is moreover liable to rather rapid deterioration. It always requires very careful setting up, and a judicious choice of wood for the pivot; it needs to be always submerged and must not be used with salt water or water laden with grit.

To avoid these disadvantages ball-bearings may be used in connection with it, in which case the friction is considerably reduced in comparison with that ordinarily developed.

The regulating device for mixed turbines generally consists of a cylinder which is raised or lowered at will in the interval left between the two crowns. This device produces throttling of the jets at the outlet of the fixed crown, with the result that the efficiency is appreciably reduced when the amount of the opening is diminished.

In order to avoid this defect to some extent, the vanes are sometimes provided with a series of partitions which divide the full depth of the entrance into several compartments. No contraction is then produced when the edge of the gate is brought to rest just flush with one of the partitions, but with an intermediate position of the edge contraction is not avoided.

To sum up, mixed turbines possess advantages over other turbines as regards their reduced dimensions, and in the ease and cost of their erection; but the shape of the vanes makes their construction rather difficult; their pivots and gates are relatively inferior; they cannot be used for high falls on account of the excessive rotational speed that would result; they are not well adapted to variations of flow, and while not possessing any superior efficiency at full gate, have a rather inferior efficiency with reduced flow, in consequence of the type of gate used with them.

51. Description of some mixed turbines. *Hercules turbine.* This is one of the best known of the American mixed turbines. Fig. 42 gives a general idea of this turbine, and enables its construction and method of installation to be observed.

The machine which is placed in the water chamber ending the penstock, is supported by a platform or an arch built over

the tail-race, level with the crown T which corresponds to the base of the fixed blades. Round this same crown are fixed the guide-vanes I. The moving blades form part of a cylinder, level with the guide-blades at the top but projecting below the level T with spoon-shaped surfaces. These blades are detachable, which very much facilitates their construction and allows them to be worked with all the finish and polish desirable to reduce the friction of the water against their sides to a minimum. The ease with which the turbine is taken to pieces is also a decided advantage as regards renewals of damaged blades.

These blades are fixed by means of bolts to the blade-carrier M which is keyed to the axle of the turbine. To ensure complete rigidity in the fastening of the blades, each blade has a tongue cast on its upper edge which fits into a circular groove turned in the boss of the carrier in its lower face. This tongue is drawn tightly into place by the corresponding bolt. Further, the lower edges of the blades are connected by a steel ring f to which they are riveted, so that the whole forms a rigid structure.

The gate V between the two crowns is simply a cylinder capable of being moved upwards or downwards by means of a controlling mechanism. For this purpose, the cylinder is hung from two racks c, which gear with two pinions p, mounted on an axle G, carried by bearings cast with the upper part of the machine. This mechanism is actuated by means of two bevel wheels R situated to the right of the shaft G. The racks are protected from splashes of water by casing on the top of the machine reaching to the pinions.

The cylindrical gate is provided with claws T, which form a ring of teeth round the lower edge sliding between the fixed blades, these serve to guide the water entering the moving crown, and the contraction which would otherwise be produced by the relatively sharp edges of the gate is thereby avoided.

It will be seen in the figure that three ribs are cast on each of the blades; each of these ribs forms a sort of partition

which divides the height of the entrance into compartments corresponding to the different positions which one may desire to give to the gate.

If for example the gate be raised to exactly the height of the middle rib, this rib will perform the same function as does the boss of the carrier when the gate is wide open, and the upper boundary of the jet will be so guided by the curved surface of the rib, that contractions tending to reduce the efficiency will be avoided.

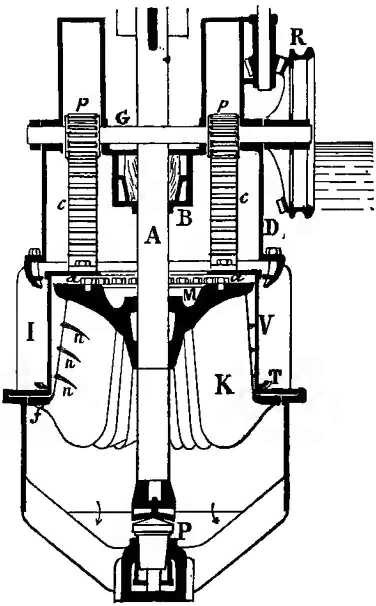

Fig. 42. Hercules turbine.

For this device to be of use, the lower edge of the bell must be stopped exactly in line with one of the partitions, for if it is between two ribs the contraction is not eliminated. Gates which are simple in design are always defective from this point of view.

The vertical axle rests on a drowned hard-wood pivot. A cast-iron box with arms connecting it with the sides of the lower cylinder, contains the footstep into which is driven with a conical fit, the hard-wood cone P on which the axle pivot rests. This pivot is of cast-iron having a socket on its upper side into which is fitted the end of the axle, and its lower side ends in a hollow spherical surface which is pivoted on the convex surface of the hard-wood pin.

The upper portion of the axle is guided by a bushing B fixed beneath the upper end of the cylinder D into which the gate fits when raised.

Brault turbines. Large numbers of turbines have been constructed on the same principle as the Hercules turbine, but differing slightly in arrangement and constructional detail.

Thus in the Brault turbine, the vanes of the moving crown, instead of being independent, are cast in one piece. This construction is obviously cheaper than the other, but is not so well adapted for high finish.

Laurent turbines. Laurent Bros. and Collot have brought out a different arrangement for the cylindrical regulating gate; instead of being interposed between the two crowns, it is placed outside the guide-crown. This allows the cylinder D which is no longer necessary to be done away with, because, as the gate surrounds both crowns, these latter can have one cover common to both, and there is a water-tight joint between this cover and the cylindrical gate; the total height of the motor is therefore reduced by this amount and the construction is thereby simplified.

Further, the gate being outside the distributor, it is no longer necessary to leave so much clearance between the two crowns; however, as previously stated, there is not much advantage in this, for the space necessary for the gate to pass does not appear to be in any way injurious from the point of view of the motion of the currents inside the turbine.

The footstep bearing instead of resting in a box closed at the bottom, may be placed, as in this turbine, in a bottomless

socket in which it can slide, its height being adjusted from a lower cross-bar which enables wear to be taken up.

Vigreux turbine. Darblay and Son manufacture a turbine designed by Charles Vigreux, in which the gate system used is quite different in principle from any of the foregoing. It has been shown that the cylindrical bell gate prevents the maximum efficiency being maintained as the flow is changed, so that the fall is not properly utilised when the turbine is not working at full load. This is because the cylindrical gate acts on all the guide passages together reducing the flow in every one of them at the same time, thereby causing contractions and eddies which lower the efficiency.

This defect is avoided if, instead of partially shutting all the passages, a gate is so arranged to close them completely but one at a time, according to requirements. Under such conditions, the flow could be varied to the extent desired without loss of efficiency, since the passages left open would be working at full bore.

To secure this result, the guide-crown is formed of a number of fixed vanes constituting the distributor proper, and an equal number of hinged vanes placed between them, which may be shut to like a door closing the whole height of the passage. The hinged vanes are mounted on vertical axes arranged round the guide-crown. Each of these spindles has at its upper end a small horizontal lever with a pin at its extremity, these pins work in a circular groove in the rim of a wheel. This latter is arranged so as to be able to turn round the vertical axle.

This wheel has as a matter of fact two grooves, one on its upper face and one on its lower face, and the pins of the valves situated on each half of the circumference engage with the two grooves respectively. Each groove is formed of two semicircles of different radii, so that as the wheel is turned, the levers controlling the valves are guided radially or tangentially, closing or opening the valves.

As moreover the semicircular grooves are arranged in an opposite manner on the two sides of the wheel, the passages

are always closed in pairs, opposite ends of the same diameter being worked together, so that symmetry of action of the water in the turbine is preserved.

The regulating wheel is controlled by means of a toothed wheel attached to its upper side, and geared with this wheel is a pinion whose vertical spindle is actuated by a hand-wheel.

The whole of this mechanism is contained in a water-tight bell surmounting the turbine. The axle of the moving crown passes through two bearings, one in the end of the bell, and the other higher up. The turbine is suspended from the latter by means of a sleeve carried on a ball-bearing. This bearing being placed right above the water gives complete satisfaction.

The Leffel turbine. In America the Leffel turbine is one of the most widely used.

Its distinguishing feature is that the moving crown is really two crowns superposed; the upper crown is so designed as to direct the water towards the centre and is therefore an inward-flow turbine; the lower crown has vanes possessing double curvature which discharge the water parallel to the axis as in parallel-flow turbines. There seems to be no particular advantage in this arrangement.

Another peculiarity of this turbine is that the guide-blades are moveable and can oscillate about axes parallel to the turbine axle; they perform the function of guides and also of regulating device, diminishing or increasing the flow according to their inclination.

The Bookwalter and Tyler turbine. Fig. 43 represents a turbine of this type as modified by Bookwalter and Tyler. The two crowns eg and $i'g'$ are separated by a cylindrico-conical partition dd'. The guide-vanes are shown at O; these vanes serve as gates and for this reason are hinged between two discs B and F; they are worked together by an annular ring H which has a toothed sector U′ actuated by the pinion X′ and the hand-wheel W′.

With the gates full open, the efficiency lies between 65 and

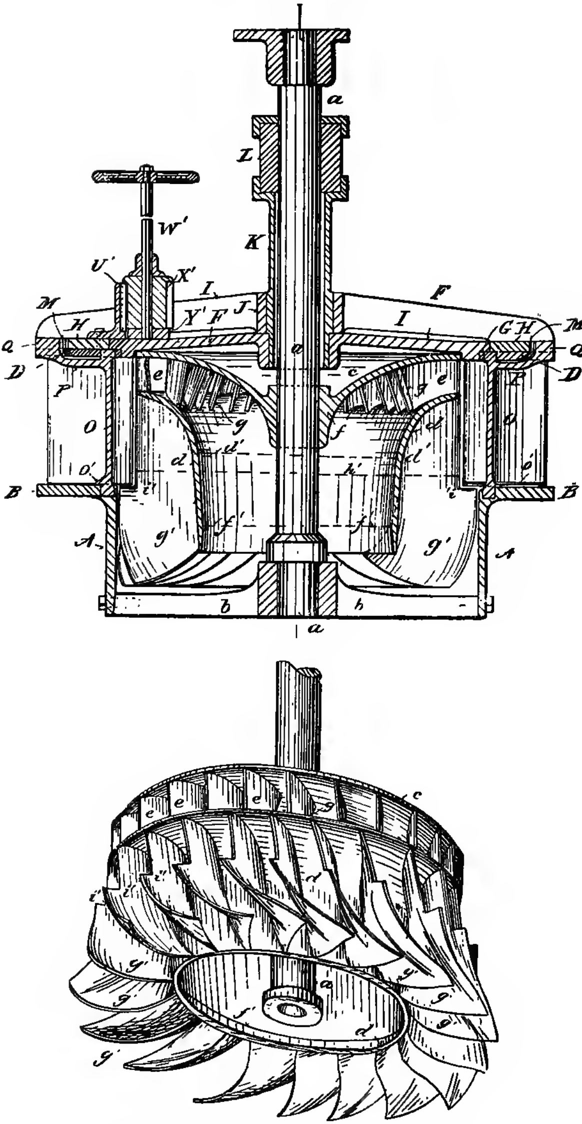

Fig. 43. Bookwalter and Tyler turbine.

74 per cent., but it falls to 60 per cent. when the opening is reduced by one half.

Ridson turbine. Fig. 44 is a sectional elevation and plan of a Ridson turbine. In this turbine the guide-blades D are flat; the moving vanes G possess double curvature and are spoon-shaped at the end.

The gate is a cylinder V which is hung by a rack *c* from a piston *d*, which is concentric with and can slide along the axle. This gate is guided very accurately in the slots of three special guides J, and as a result the clearance left between the gate cylinder and the moving crown is reduced to an exceedingly small amount. The rack supporting the gate is actuated through a pinion *p* and a train of gearing.

The weight of the gate is almost exactly balanced by the pressure of water from the fall acting on the lower side of the piston *d*.

The moving crown is completely protected from water on its upper side by the case B which carries a sheath M which in turn acts as guide to the axle A. This axle works on a wooden pivot P which is always kept under water.

Brenier and Neyret turbine. Mixed turbines may also be made with horizontal axes. The Brenier and Neyret turbine is one of the most interesting of this class. The guide-crown is fixed inside a spiral envelope into which the water enters by a tube on the lower side; the current of water is distributed over the whole area of the guide-crown, penetrates into the moving crown placed concentric with and inside of the guide-crown and then escapes in a direction parallel to the axis by an opening constructed in one of the vertical sides of the case.

In fact the water follows a path which is just the opposite of that taken by air in a centrifugal fan, in which the air is drawn in through an opening at the side and is driven out through a pipe which is a continuation of the envelope spiral.

In the turbine now under consideration, the discharge orifice is prolonged by a bent pipe which dips into the tail-race. The barrel is cast in two parts whose flanges are bolted

together; the bearings which support the shaft are cast one with one side of the barrel and the other with the bent discharge pipe.

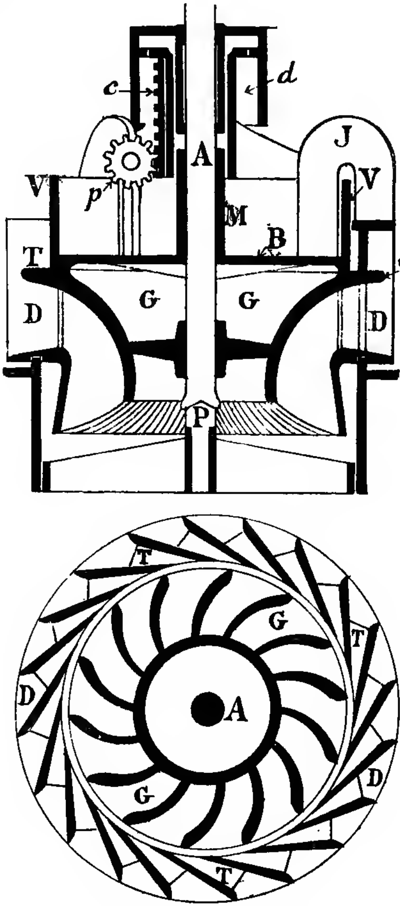

Fig. 44. Ridson turbine.

Such a turbine is generally used for driving dynamos with

horizontal axes, which may be driven directly through an elastic coupling.

The cylindrical gate used is just like those employed with the vertical turbines; it is worked from a hand-wheel by means of a train of wheels.

Coupled Turbines. Mixed turbines are generally used in America for large falls. For low falls of large volume, two turbines coupled together on the same shaft may be used.

This arrangement has been applied to the Leffel turbine described above. The two moving crowns mounted at extremities of the same shaft are enclosed in a case of sheet metal whose two ends are connected with the penstock; after having passed through the two turbines, the water flows through a common discharge pipe connected to the middle of the case between the two motors.

This pipe dips into the tail-race after the manner of that used in the Girard siphon turbine. As all siphons must first of all be primed, the discharge pipe has a tube on its side communicating through a valve with the head-race to effect this priming.

The regulating gates are arranged in the manner indicated above for vertical turbines.

Turbines may be coupled together on the same shaft in the contrary manner, that is to say the water may enter through the centre of the common envelope, while the discharge takes place from the ends through two distinct pipes each dipping into the tail-race. The Jonval device is generally used with horizontal turbines. It has the special advantage that it enables the turbines to be installed on the floor of the engine-room at an appreciable height above the tail-race, without the available fall being diminished.

High Falls. It has been shown that mixed turbines, which are particularly in favour in America, are not at all superior to French turbines as regards efficiency. Moreover these turbines are not well adapted for high falls of small volume, in consequence of the enormous speeds they would attain under such circumstances.

On the contrary the ordinary parallel-flow turbine is very suitable for high falls. It is advisable, however, as has been pointed out (§ 47), not to feed the turbine over the whole of its periphery but only over a portion. Then the diameter of the turbine and the orifices will be very large; hence the speed of rotation will be reduced on the one hand and the passages through which the flow takes place will be less liable to obstruction on the other hand.

The installation of this turbine is a very simple matter; its upkeep moreover is rendered easier because the moving crown is quite accessible; the system is also to be recommended for small installations in mountainous districts where a small and not highly skilled staff is employed.

52. The Pelton wheel. Years ago, jets of water at high pressure were used to drive wheels by striking vanes fixed on the circumference and imparting thereto their kinetic energy. The water was discharged from a tapered spigot or nozzle; some mills used to be driven in this manner.

The wheel vanes were spoon-shaped and the water struck them violently near their extremities, acting on them in the same way as air on the sails of a windmill. Obviously with such elementary devices high efficiencies could not be obtained, in fact the efficiency of such systems did not exceed 33 per cent.

An American maker named Pelton has perfected this old system by replacing the hollow vanes with a more rational form of bucket, which enable efficiencies to be obtained equal to that of turbines.

This wheel, which has been widely used in America, is especially suitable for the utilisation of very high falls; it receives the kinetic energy of the mass of water, which has a very high speed, and can consequently develop considerable power.

It consists of a cast-iron wheel, on the rim of which are secured a number of hollow buckets, in the middle of each of which there is a sharp V-shaped edge with sides curving smoothly to right and left (fig. 45).

The jet of water discharged by the nozzle is divided into two parts by the V, and the two sheets are turned back by the curves of the bucket. The water, after having given up its kinetic energy to the wheel in consequence of this deviation, falls gently behind the bucket.

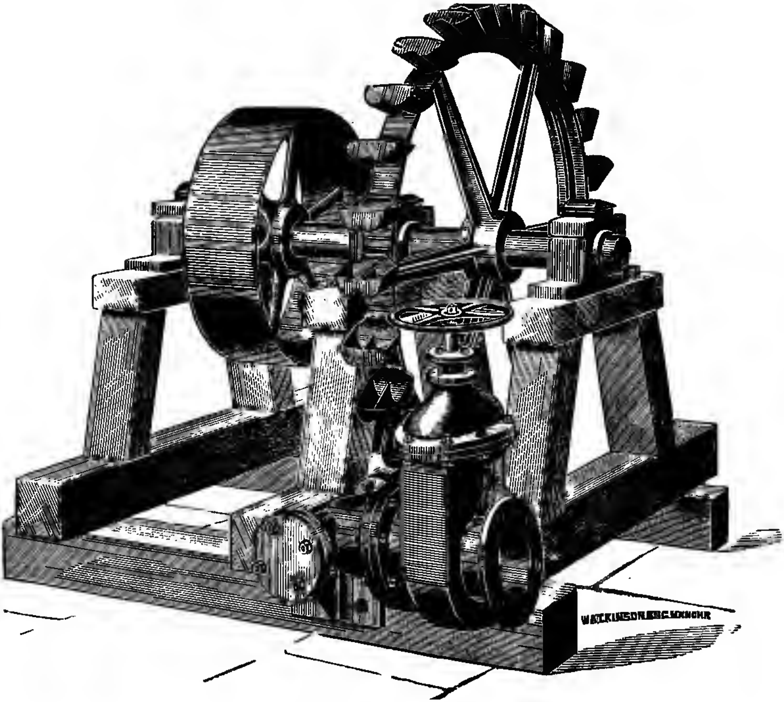

Fig. 45. Pelton wheel.

The buckets which receive the impulse of the jet and transform it into mechanical work are the essential part of the apparatus, for on the degree of perfection of the shape of these buckets the value of the efficiency depends. This efficiency may exceed 80 per cent. in a well-made wheel.

Fig. 46 shows a section through a bucket of a Pelton

wheel on a plane parallel to the axle of the wheel. It will be seen that the bucket is divided into two compartments by the sharp-edged partition C. The general form of the transverse section is like that of a figure 3. The round jet of water escaping from the nozzle A is divided into two by the edge C, and these two jets glide laterally to the right and to the left over the curved surfaces of the two buckets and escape in two sheets N, whose direction is practically parallel to that of the axis SS′ and in the opposite sense to that of the main jet.

The efficiency depends on the ratio between the speed of the jet and the peripheral speed of the wheel.

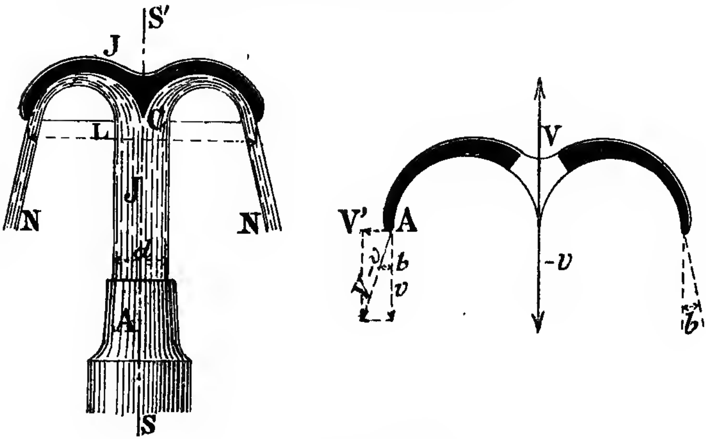

Fig. 46. Pelton bucket.

If H be used to denote the height of the fall which determines the velocity of the water spurting from the nozzle A, the velocity V of the water at the outlet will always be given by:

$$V = \sqrt{2gH}.$$

The speed with which the water travels in the inside of the bucket—or, briefly, the relative velocity—is obviously the difference between the absolute velocity V and that of the

bucket v, since the bucket travels in front of the jet; therefore this relative speed is:

$$w = V - v.$$

The motion of the bucket is due to the absorption of the kinetic energy of the water by the wheel; consequently this kinetic energy would be wholly transformed into mechanical motion of the wheel, and the efficiency would be a maximum if the absolute velocity V' at the outlet of the buckets were zero.

Let b be used to represent the angle between the direction of the relative speed of the water at the outlet with a line parallel to SS′; a parallelogram of velocities can now be constructed at A, having as diagonal the relative speed $V - v$, and for the two sides the circumferential speed of the wheel v and the absolute velocity V' of the water at the outlet.

In one of the triangles thus constructed we have the geometric relation holding:

$$V'^2 = (V - v)^2 + v^2 - 2v(V - v) \times \frac{1}{\sqrt{1 + i^2}},$$

where i is used to denote the inclination of the relative speed $V - v$ to the direction parallel to SS′.

But since, as has already been stated, the inclination of the jet is very small, the angle b being very little different from zero, i may be taken as zero, and the previous relation simplifies to:

$$V'^2 = (V - v)^2 + v^2 - 2v(V - v);$$

or again, using w to represent $(V - v)$:

$$V'^2 = w^2 + v^2 - 2vw.$$

Now in this form the square of the quantity $(w - v)$ will be recognised, whence we may write:

$$V'^2 = (w - v)^2 = [(V - v) - v]^2,$$

or:

$$V' = (V - v) - v.$$

The condition for maximum efficiency will be obtained therefore on putting:

$$V' = V - v - v = V - 2v = 0.$$

From which we may finally deduce;

$$v = \frac{V}{2}.$$

Thus the efficiency will be a maximum when the velocity of the rim is half the speed of the water at the outlet of the nozzle A.

It will be observed that the buckets fixed on the periphery of the wheel cut through the jet with their edges as they come into the position in which the jet acts on their faces From consideration of this fact it will be seen to be inadvisable to place the buckets too closely together, or the jet would be broken more often than necessary, and the friction resulting would diminish the efficiency.

On the other hand, they must not be too widely spaced, or the water would be able to get past the wheel in the interval between two consecutive buckets without being checked by either of them, and consequently without doing any useful work.

The rule to be followed in determining the correct number of buckets is to find the minimum number for which the spacing does not exceed the limit beyond which loss of water would take place.

In order that there may be no loss of water it is necessary that when the lower edge of the bucket in rising is just about to leave the outer edge of the jet, the next bucket shall have descended into a symmetrical position in relation to the jet of water so that it just embraces the whole jet as the former bucket is about to lose a portion.

If h be used to represent the perpendicular distance from the centre of the wheel to the axis of the jet, d the diameter of the latter, and R the radius of the wheel to the extremity of the buckets, it may be demonstrated geometrically that the number of buckets is given by the equation:

$$n = \frac{\pi}{\sqrt{1 - \frac{\left(h + \frac{d}{2}\right)^2}{R^2}}};$$

in this formula $\left(h+\frac{d}{2}\right)$ represents the perpendicular distance from the centre of the wheel to the outer boundary of the jet; putting:

$$h+\frac{d}{2}=l,$$

the above equation becomes:

$$n=\frac{\pi}{\sqrt{1-\frac{l^2}{R^2}}}.$$

Obviously if l be made equal to R, that is if the jet be placed in such a position that its outer boundary is tangential to the outer circumference of the buckets, then we should have:

$$n=\frac{\pi}{\sqrt{1-1}}=\frac{\pi}{0}.$$

That is to say, the number of buckets would be theoretically infinite, consequently l must always be given a value less than R and the number of buckets for a given radius will be less in proportion as the jet is placed nearer the centre of the wheel. But there is also an upper limit beyond which if the jet were moved it would strike the inside of the wheel in the interval between two buckets; this would occur if the inner boundary of the jet were nearer to the centre of the wheel than the inner circumference of the buckets. To sum up, the jet ought to lie wholly between the inner and outer circumferences of the wheel.

It is necessary moreover that the water should leave the buckets in thin sheets so that it will not strike the backs of the following buckets.

It will readily be understood that the Pelton wheel is especially suitable for small quantities of water and high falls. The speed which depends on h is therefore always very great; for small machines it may be as high as 5000 turns per minute in the case of falls of 1000 feet; for motors of 500 horse-power the revolutions would be 400 per minute; hence it is necessary that these wheels should be mounted with great

care and should be perfectly balanced, in order to avoid shocks and vibrations which would be prejudicial to the efficiency and durability of the wheel.

Larger volumes of water may be used by projecting the water from several nozzles at once; Pelton wheels have been made with as many as five nozzles. In this case the water pressure pipe ends in several nozzles connected together beneath the wheel in an arc concentric with the circumference; the first jet is horizontal and the succeeding ones are more and more elevated so as to throw the jets more nearly vertical.

The wheel is thus supplied over a more or less considerable portion of its circumference and several buckets are under the action of the jets at the same time.

Further it is more easy to regulate the flow when several jets are used, for all that is required is to close entirely or open fully some of the nozzles. Each working nozzle is then always full open, and the jet never being throttled the efficiency is in nowise affected.

In the case of a single nozzle, the flow is varied by throttling the outlet orifice to a greater or less extent, by means of a conical baffle which is brought nearer to or taken farther from the inner sides of the nozzle which are coned concentrically with the baffle. With this arrangement it is difficult to maintain the regular form of the jet, and it is more or less broken on striking the bucket, the efficiency suffering accordingly.

In well constructed Pelton wheels the efficiency is always high and comparable with that of the best turbines. Moreover these wheels have the great advantage of developing considerable power with very restricted dimensions. Thus a wheel 2 feet in diameter may develop 125 horse-power at 1000 revolutions per minute under a fall of 650 feet. Evidently therefore it is the high speed of these motors which renders them so well adapted for very high falls.

53. The hydraulic ram. Hydraulic wheels and turbines are engines which receive the energy of waterfalls

and transform it into useful work. But a useful result may be obtained, without the aid of these motors, by simply using the kinetic energy of a stream to raise a portion of the water to a higher level.

This operation is performed by the *hydraulic ram* whose invention is due to Montgolfier. This apparatus consists of a pipe *cd* connecting a reservoir filled with water to the

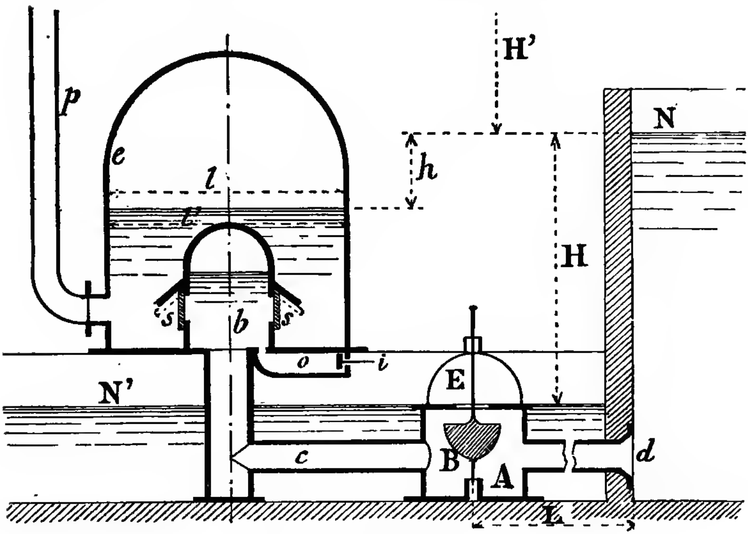

Fig. 47. Hydraulic ram.

level N with a bell *b* which is itself contained in a reservoir *e* (fig. 47). From the latter the pressure pipe *p* leaves which discharges into the higher reservoir to which the water is to be raised.

In the course of the pipe *cd* there is a valve-box A whose cover E is pierced with an orifice which can be opened and closed alternately by the action of the valve B. Between the box A and the orifice of the pipe *cd* a cock is interposed which is not shown in the figure and which serves to start or stop the working of the apparatus at will.

To complete the description of the apparatus it must be added that the bell b has two openings in the sides which are provided with automatic non-return valves s.

When not working, the valve B remains at the bottom of its travel, for it is only subject to the action of its own weight. When the cock is opened to start work, a current of water commences to flow from the reservoir N towards the valve-box opening E through the pipe cd. This water flows through the orifice E into the tail-race N′, and starting from rest, it accelerates until its velocity attains that corresponding to the height H of the fall.

Now, as we have seen in the former part of this book, the resultant pressure exerted on a disc placed in a current of water, has for its expression:

$$R = d \times A \times \frac{V^2}{2g} \times K,$$

and this equation can be used in the case of the actual valve B, on denoting the area of the upper surface of the valve by A, the density of the water in lbs. per cubic foot by d, and by K, a coefficient which depends on the contraction and on the horizontal section of the current in the box A, that is the area of the end of the box itself.

As moreover this resultant acts in the direction of the current, namely from below upwards, when a sufficient velocity V is attained, this action will overcome the weight of the valve, raising it and clapping it on its seat E.

At this instant the current in the direction of E will be sharply checked, but, in consequence of the kinetic energy which the moving mass of water possesses, it will strike against the sides of the bell b and with its ram-like blow will open the valves s. Therefore some of the water that entered the bell will rush into the reservoir e. Simultaneously, the air trapped both in the bell and in the reservoir, will be compressed.

When the force of the blow is spent, the clack-valves s shut, and the air in the bell b expanding, causes the water to flow back from b towards A. The vacuum following this expansion causes a certain quantity of air to be sucked into the

bell b through the valve i and the orifice o. The valve B is dragged downwards, as the result of the same oscillation, the orifice is reopened and causes a fresh flow of water from the source to the tail-race and the cycle of operations is repeated in the same fashion.

It will be seen what function is fulfilled by the air confined in the bell and in the reservoir. The former serves to deaden the shock of the ram blows like an elastic cushion placed between the water and the end of the vessel; the second causes a regular and continuous motion of the water in the rising pipe p. In fact the water on leaving the bell first collects in the reservoir and the level rises from l' to l, the air being compressed to a greater pressure than that of the column p, then the air expands and steadily forces the water into the column.

If we denote the diameter of the outer surface of the valve by D, that of the orifice E by D′, the distance between the valve when at the bottom of its travel and its seat by e, and finally, the thickness of the sheet of water between the outer edge of the valve and the inner edge of the orifice by l, we can compare the flow of water between the valve and its seat to that which would be produced in a cylindrical nozzle of diameter e and length l, this last quantity being defined by the relation:

$$l = \frac{D - D'}{2}.$$

Now, we know that in a cylindrical pipe, the jet contracts and gives rise to a loss of pressure equal to one-third of the head H or $\frac{H}{3}$. The effective head causing the velocity of the sheet of water is therefore reduced to $\frac{2H}{3}$ and we have for the speed of the water at the outlet:

$$V = \sqrt{2g \times \frac{2H}{3}} = \frac{2}{3}\sqrt{3gH}.$$

Moreover the valve of the ram may be likened to a plate placed in a cylindrical current, as stated above.

The effective pressure R, which tends to raise the valve, is proportional to V and the valve begins to close when this speed is such that:

$$R = d \times A \times \frac{V_1^2}{2g} \times K = P,$$

where P is the weight of the valve.

From this may be deduced:

$$V_1 = \sqrt{\frac{2gP}{dAK}}.$$

From this instant, R keeps on increasing and exceeds P so that the valve is closed with a velocity depending on the accelerating force $(R - P)$. Under the action of this force, the speed increases from zero to a maximum V_2 which is reached at the instant when the valve strikes on its seat; assuming the mean speed to be $\frac{V_2}{2}$, we have the relation:

$$e = \frac{V_2}{2} \times t,$$

or:

$$t = \frac{2e}{V_2}.$$

But the relation between the accelerating force and the mass accelerated is:

$$(R - P) = ma = m\frac{V_2}{t} = \frac{P}{g} \times \frac{V_2}{t},$$

whence:

$$V_2 = \frac{(R - P)\, g \times t}{P} = \frac{2e}{t}.$$

Therefore:

$$t = \sqrt{\frac{2Pe}{g\,(R - P)}}.$$

Now let the section of the pipe *cd* of the ram be *s* and let S represent the horizontal section of the valve-box. Then between the velocity of the water in the box A and the velocity V′ of the water in the pipe, the following relation holds:

$$\frac{V'}{V} = \frac{S}{s},$$

or:
$$V' = V \times \frac{S}{s}.$$

If L be used to represent the length of the pipe *cd*, the mass of water contained in it will be equal to the weight divided by the gravitational constant g, namely:
$$m = \frac{d \times s \times L}{g}.$$

Now this mass starts from rest and acquires a velocity V′ by the time the valve is closed. This mass is put in motion by the accelerating force due to the height of fall H, that is:
$$F = d \times s \times H.$$

It follows that under this force, the mass of water considered will have an acceleration j which according to the laws of mechanics is expressed by:
$$F = mj,$$

whence:
$$j = \frac{F}{m} = \frac{d.s.Hg}{d.s.L} = g \times \frac{H}{L}.$$

As, moreover, the velocity acquired as the result of uniform acceleration due to a constant force is equal to the product of the acceleration by the time during which it acts, we have:
$$V' = jt',$$

whence:
$$t' = \frac{V'}{j} = \frac{LV'}{gH}.$$

The time t' is that which elapses from the instant that the valve is opened until it is closed again. We have already considered the value t of the time which the valve takes in closing. Hence the time $(t' - t)$ will represent the interval of time during which the valve remains quite full open.

These considerations enable us to estimate the volume of water passing during the time t', which, it must be borne in mind, is divided into two parts $(t' - t)$ and t.

During the first period, the average value of the velocity of the water in the vessel A is:
$$\frac{0 + V_1}{2} = \frac{V_1}{2};$$

during the second, the speed changes from V_1 to V_2, and its average value is:

$$\frac{V_1 + V_2}{2}.$$

Using Q' and Q'' to represent the volumes of water delivered in each of these periods respectively, we have:

$$Q' = \frac{S . V_1 \times (t' - t)}{2} = \frac{S}{2}\sqrt{\frac{2gP}{dAK}} . (t' - t)$$

and:

$$Q'' = \frac{S}{2} . (V_1 + V_2)\, t.$$

But it has been shown further back that:

$$\frac{V_2 t}{2} = e.$$

Whence we have:

$$Q'' = S \times \left(\frac{t}{2} \times \sqrt{\frac{2gP}{dAK}} + e\right).$$

The volume of water delivered in the total time t' is therefore:

$$Q = Q' + Q'' = S \times \left(\frac{t'}{2} \times \sqrt{\frac{2gP}{dAK}} + e\right).$$

The time t' corresponds to the maximum velocity V'; this depends solely on the height of the fall, if friction be neglected:

$$V' = \sqrt{2gH},$$

whence:

$$t' = \frac{L}{gH} . V' = L \times \sqrt{\frac{2}{gH}}.$$

Finally therefore the value for Q is:

$$Q = S \times \left(L \times \sqrt{\frac{P}{H . d . A . K}} + e\right).$$

This mass of water possesses kinetic energy which is employed in forcing a certain volume q of water into the reservoir C. Now the mechanical work done by a fluid is equal to the volume moved multiplied by the pressure against which it moves. This latter is represented by a column of water of height $(H' + h)$ to which must be added the loss of

head z due to friction in the pressure pipe. The useful work done at each blow of the ram will be therefore :

$$Tu = d \,.\, q \times (H' + h + z).$$

Neglecting losses, this work must be equal to the kinetic energy of the mass of water in motion, and this latter is expressed by half the mass multiplied by the square of the velocity ; therefore, using D to rèpresent the diameter of the pipe cd :

$$Tu = d \,.\, q \times (H' + h + z) = \frac{d \,.\, \pi \,.\, D^2 \,.\, L}{4} \times \frac{V'^2}{2g},$$

from which :

$$d \,.\, q \times (H' + h + z) = \frac{d \,.\, \pi \,.\, D^2 \,.\, L \,.\, V'^2}{8g},$$

and finally :

$$q = \frac{\pi \,.\, D^2 \,.\, L \,.\, V'^2}{8g \times (H' + h + z)}.$$

This equation shows that the volume of water raised per stroke of the ram is proportional to the size of the ram, to the square of the maximum speed of the water and inversely as the height of the rise to the reservoir C.

The energy employed is equal to the product of the total volume moved, by the pressure due to the height of the total fall H, that is to say :

$$Tm = d \,.\, Q \,.\, H.$$

Hence the efficiency will be obtained by taking the ratio :

$$\frac{Tu}{Tm}.$$

The expression thus obtained for the efficiency shows that it is proportional to the square of the diameter of the pipe cd and to the maximum speed V ; it decreases if the section of the box A be increased, and also with the weight P of the valve and its travel e ; on the contrary it increases with the area of the valve face.

The efficiency of a well constructed ram is about 65 per cent. on the average, it varies between 50 and 70 per cent. The number of strokes per minute is given by the equation :

$$n = \frac{60}{t' + t''},$$

where t'' is the time in seconds during which the valve remains shut and which must always be very short, say about half a second.

The capacity of the regulating reservoir C is estimated from the fact that it receives per second a volume $\frac{q}{t''}$, while the volume delivered through the rising column p in the same time is only $\frac{q}{t'+t''}$; hence the excess stored during the time t'' is:

$$q' = \left(\frac{q}{t''} - \frac{q}{t' + t''}\right) \times t'' = \frac{qt'}{t' + t''} = q\left(\frac{1}{1 + m}\right) \text{ say.}$$

As a consequence the height of the water in the reservoir oscillates between two levels l and l'. The air trapped in it is therefore alternately compressed and expanded. On imposing the condition that the difference between the extreme values of the air pressure shall be a fraction $\frac{1}{N}$ of its average value, lying between $\frac{1}{50}$ and $\frac{1}{100}$, the following relation giving the capacity of the reservoir is obtained:

$$C = \frac{\pi . D^2 . L}{4} \times \frac{2H \times N}{3(1 + m) \times (H' + h + z)},$$

which shows that the regulating reservoir ought to be proportioned to the volume $\pi D^2 L$ of the body of the ram, to the height of the fall H, and in the inverse ratio of the height to which the water is to be raised.

Although the hydraulic ram is not a motor intended for the development of motive power, it has been deemed well worth while to devote a paragraph to it, for it forms a valuable piece of apparatus by which country houses and mansions situated near a little stream or brook can be supplied with water very simply and at a comparatively low cost; for the water may easily be raised to such height as will be convenient for distribution over the whole premises. In the author's opinion, it is desirable that this device which is capable of rendering such excellent service, should be used much more often than appears to have been the case up to the present.

CHAPTER V.

THE CONSTRUCTION OF A WATERFALL.

54. Preliminary operations. Hydraulic motors and devices designed to receive the energy of waterfalls and transform it into mechanical work have now been considered at some length. But waterpower, although called a natural source of power, cannot be used straightaway without forethought and outlay.

It is now necessary therefore to study the construction of falls as commercial undertakings. This question embraces the construction of supply canals and tail-races in connection with the installation and the necessary regulating contrivances.

Let us suppose that part of a stream is available on which it is intended to create a fall to supply an installation. We say part of a stream, and not a stream, because generally the length with which one has power to deal is limited. Obviously of course, penstock and tail-race may not be constructed on adjoining properties and the latter canal must discharge into the water-course within the down-stream limits of the property belonging to the installation. Moreover it is equally essential that the erection of a retaining dam shall not occasion any eddies or rise of level up-stream beyond the confines of the same property.

Thus in principle at any rate, the projected installation ought not to modify in any way the state of the stream above the boundary line of the property owned, nor ought it of course to encroach on neighbouring property.

At the outset of the project it is necessary to prepare a large scale plan of the portion of the water-course over which full power has been acquired. This topographical plan should have marked on it the lines of constant level in the neighbourhood of the banks; in brief it should be a contour map.

It is then necessary to draw the longitudinal profile of the stream along its centre line; this should also be repeated for ordinary levels and for times of low water.

In considering these points, the principle established earlier when dealing with the flow of water in open canals should be called to mind. In this connection it was mentioned that when instead of dealing with a canal, an actual stream was to be dealt with, the cross-section as well as the slope of the bed varied from point to point throughout the course. Hence the motion of the water is not uniform and the speed is different in successive sections.

Another fact is that the surface of the water is not parallel to the bed and its level may vary in such a manner that the depth of water may increase or diminish from one section to another.

Being given the longitudinal profile of the river bed, cross-sections should be drawn at approximately equal intervals along its course, but taking care to choose them at highest and lowest points in the profile. It will easily be seen from examination of this profile how the slope of the bed varies in the different intervals between consecutive sections. The difference in height of two consecutive sections of the bed divided by the distance separating them, will give the slope in each interval. If this slope exceeds about 4 per cent. the stream will be classed as a torrent.

55. Estimation of flow. The first question to be solved is the estimation of the volume delivered by the stream. It is necessary to consider the flow both during ordinary floods and for low water.

Of course the delivery Q is the same at all sections, but if this be calculated for a single section only, one is liable to

obtain a value either too large or too small. The best plan is to choose a certain number of the most characteristic transverse sections and to take the mean of the results obtained on applying the formula to these various sections.

Although this problem has not been directly treated in this book, it follows at once from the calculation given in connection with the finding of the difference in water level between two bounding sections of a portion of an irregular stream (§ 20).

This difference of level was given by the relation:

$$y_3 - y_2 = \frac{Q^2}{2g}\left(\frac{1}{S_3^2} - \frac{1}{S_2^2}\right) + \frac{Q^2 \times b_1}{2} \times \left(\frac{P_2}{S_2^3} + \frac{P_3}{S_3^3}\right) \times d.$$

In this formula, $y_3 - y_2$ represents the difference of level between the two sections considered; Q represents the volume of the stream; S_2 and S_3 the areas of the sections obtained from the transverse profiles at the corresponding points; b_1 the Darcy coefficient which applies to the resistance resulting from the friction of the water against the banks; P_2 and P_3 are the wet perimeters of the two sections, and d is the distance between them.

Actually we know the longitudinal profile and consequently the quantity $y_3 - y_2$ which we will denote by y for brevity; it is the value Q that is required. Let us therefore put the foregoing expression in the form:

$$Q^2 = \frac{y}{\frac{1}{2g} \times \left(\frac{1}{S_3^2} - \frac{1}{S_2^2}\right) + \frac{1}{2}\left(\frac{P_2}{S_2^3} + \frac{P_3}{S_3^3}\right) \times b_1 \times d.}$$

Now the factor $\frac{1}{2}\left(\frac{P_2}{S_2^3} + \frac{P_3}{S_3^3}\right)$ represents the mean of two quantities relating to the two sections considered so that we may consider a section S, lying between the latter two and for which:

$$\frac{P}{S^3} = \frac{1}{2}\left(\frac{P_2}{S_2^3} + \frac{P_3}{S_3^3}\right).$$

Putting this in the principal equation, this latter becomes:

$$Q^2 = \frac{y}{\frac{1}{2g}\left(\frac{1}{S_3^2} - \frac{1}{S_2^2}\right) + \frac{P}{S^3} \times b_1 \times d.}$$

But this relation was obtained on the assumption that the sections S_2 and S_3 were very close to each other, namely, so near together that the transverse profiles between the two sections could not vary appreciably and that the slope over this short length could also be considered uniform. Now, we are practically dealing with sections distant 150 or 200 feet from each other, so that between the two sections the area S of the intermediate profiles and their perimeter P may vary to a considerable extent from one point to another in the stream; similarly with the values of b_1 we may have very wide variations.

Thus all the quantities in the term relating to frictional resistance, namely, in the expression:

$$\frac{P}{S^3} \times b_1 \times d,$$

vary continuously in the interval under consideration.

We will assume, however, that b_1 remains constant and equal to 0·000122, although actually it varies with the form and slope of the bed, for these variations are rather feeble in practice, and moreover this assumption very much simplifies the calculation.

However, the total loss of head cannot be calculated at once; it is necessary to calculate it theoretically for each length d of the course, each length being so short that the assumption made in obtaining the equation may hold. Briefly, it is necessary to estimate the sum of all the elementary fractions of the resistance over the course under consideration.

Hence one need not be concerned with the whole course at once, and it is preferable, for the sake of accuracy in successive calculations of the value of Q, not to take sections

1 and 2, but sections 1 and 3, the sections being numbered in order down the stream, say.

In this way S_2 will be taken as the area of section number 1, S_3 as the area of number 3, and S that of intermediate sections. The profiles and the different values of $\frac{P}{S^3}$ for all the intervening points are not known, but may be calculated by summation formulae such as Simpson's.

For example, let us suppose that the distance between the extreme sections considered—numbers 1 and 3—is 251 ft.* We will now denote the sections by corresponding indices in numerical order, and let us first of all calculate the S_1, S_2, and S_3. This calculation is easily carried out by splitting up the area of the transverse sections into triangles and trapeziums; similarly the wet perimeters P_1, P_2, and P_3 will be calculated from the same transverse profiles and the ratios $\frac{P_1}{S_1}$, $\frac{P_2}{S_2}$, and $\frac{P_3}{S_3}$ deduced.

Let us assume that the following results have been found:

$$\frac{P_1}{S_1^3} = 0{\cdot}0000777,$$

$$\frac{P_2}{S_2^3} = 0{\cdot}000122,$$

$$\frac{P_3}{S_3^3} = 0{\cdot}0000975.$$

We should then have, according to Simpson's formula, for the sum of all the elements of the resistances over the whole course of 251 ft. between the three sections considered:

The sum of the quantities $\frac{P}{S_3} \times b_1 \times d$

$$= \frac{251 \times 0{\cdot}000122}{3 \times 2} \times (0{\cdot}0000777 + 0{\cdot}0000975 + 4 \times 0{\cdot}000122)$$

$$= 3{\cdot}38 \times 10^{-6}.$$

* This example and the following ones are taken from Vigreux's work.

Afterwards the same calculations should be carried out for the intervals from 2 to 4, from 3 to 5, and so on, until the last section is reached. After having drawn up a list of these different values for the season of drought, a second list should be prepared for the case of ordinary floods, by calculating the areas and wet perimeters in this second case.

The expression for Q also requires the calculation of the quantities:

$$\frac{1}{2g} \times \left(\frac{1}{S_3^2} - \frac{1}{S_1^2}\right).$$

In this particular example it was found that:

$$\frac{1}{2g}\left(\frac{1}{S_3^2} - \frac{1}{S_2^2}\right) = 2{\cdot}53 \times 10^{-7}.$$

The corresponding value of Q may then be calculated.

The expression for Q embraces the quantity y, which is the difference of water level between sections 1 and 3, say:

$$y = 1{\cdot}36 \text{ ft.}$$

We have, therefore:

$$Q^2 = \frac{1{\cdot}36}{2{\cdot}53 \times 10^{-7} + 33{\cdot}8 \times 10^{-7}} = \frac{1{\cdot}36}{36{\cdot}33 \times 10^{-7}}.$$

From which may finally be deduced in the case of drought:

$$Q = 605 \text{ cu. ft.}$$

Proceeding in the same manner for the other intervals, other values of Q will be found which differ very appreciably amongst themselves, whereas obviously the flow must be the same in all the sections. But these discrepancies are explained by probable errors in levelling, and in drawing the transverse profiles, and also by actual variations in the flow, and in the form of the section of the bed which may take place during the rather long time which must be employed in successively obtaining the longitudinal profile and the various transverse profiles.

It is advisable to take into account only those results which agree moderately with one another, and to choose especially those intervals in which the transverse profiles

have no sudden changes. Then the mean should be taken of several of the results which agree most closely amongst themselves.

For the remainder of the example, we will adopt the following figures:

(1) During low water:

$$Q = 664 \text{ cu. ft.}$$

(2) For ordinary floods:

$$Q = 1350 \text{ cu. ft.}$$

56. Calculation of the New Profile. For a given stream we have been able to obtain the longitudinal profile of the surface, both for ordinary floods and for low water. The cross-sections and the longitudinal profile have enabled us on completing the assumptions to calculate approximately the flow of the stream at low water and during ordinary floods.

But we obviously do not know the new profiles which will be produced as the result of the erection of the dam. Now the longitudinal profile ought to be such that no backwash or rise of surface takes place in the first section up the stream; in a word, the level there must remain unaffected by the erection of the dam.

In order to determine what the new longitudinal surface of the water will be, we must again make use of the equation referred to above, applying it first to the two first sections:

$$y_2 - y_1 = \frac{Q^2}{2g}\left(\frac{1}{S_2^2} - \frac{1}{S_1^2}\right) + \frac{Q^2 \times b_1}{2}\left(\frac{P_1}{S_1^3} + \frac{P_2}{S_2^3}\right) \times d.$$

In this equation we know y_1 since this height must by hypothesis remain the same as before the erection of the dam. But it is evident that when the dam is built, the sections of the stream and the wet perimeters will be different from what they were before, since the longitudinal profile of the surface will have been modified. Hence we do not know beforehand the new values of S_2 and P_2.

Therefore we can only solve the above equation by the method of successive approximations. For this purpose we will first suppose that y_2 is equal to y_1, that is to say that the surface of the water is horizontal between the first two up-stream sections. We can now deduce the values of S_2 and P_2 which result from this assumption.

Let us assume that the area S_2, which was at first 91 square feet, has now become equal to 181 square feet, and that the wet perimeter P_2 has changed from 72·4 feet to 94 feet. The foregoing equation then becomes:

$$y_2 - y_1 = \frac{664^2}{2g} \times \left(\frac{1}{181^2} - \frac{1}{104^2}\right) + \frac{664^2}{2} \times 0{\cdot}000122$$
$$\times (0{\cdot}0000777 + 0{\cdot}00001575) \times 135{\cdot}2,$$

or, on making the calculations:

$$y_2 - y_1 = -0{\cdot}139 + 0{\cdot}112 = -0{\cdot}027 \text{ ft.}$$

We thus obtain a first approximation for $y_2 - y_1$, which gives at the same time a first approximation for y_2, since y_1 by hypothesis is unchanged. According to this, the section S_2 would not actually have the value 181 square feet which was the area on the assumption that the surface between the two sections was horizontal; suppose the new value for S_2 is 179 square feet and the wet perimeter is 93·9 feet.

It is then necessary to put these new values in the preceding equation when a second value for y_2 will be obtained. This process must be continued by successive approximations and substitutions until two consecutive values of $y_2 - y_1$ only differ by a negligible quantity.

Suppose that in the actual case the value found for $y_2 - y_1$ is equal to $(-0{\cdot}0295)$. This negative value indicates that there is a reverse slope between the sections 1 and 2, so that the surface rises from the origin to section number 2.

Having thus obtained y_2, the same process is carried out for the interval between sections 2 and 3, that is y_3 is first supposed equal to y_2, which enables fresh values to be calculated for S_3 and P_3 corresponding to this assumption, and the operation for this second interval is continued in the same

manner as for the first. In this way step by step the surface levels are determined for every section up to the last, and on connecting the corresponding points by a continuous line the profile of the river surface is obtained.

It is necessary to repeat the same process for ordinary floods, which gives the longitudinal profile for high water.

Evidently the crest of the dam ought to be chosen with regard to the longitudinal profile which corresponds to the lowest levels. If the line for drought is lower than that for ordinary floods, the crest of the dam should come on this line wherever the dam is being built; in ordinary floods and with average flow, a sheet of water will pass over the weir and the thickness of this sheet will be between zero and the difference of height of the two profile curves at the point where the dam is built.

57. Site of Weir. The choice of the site of the dam depends very much on the nature of the sections and the form of the longitudinal profile of the river bed. Obviously if there is a sudden change of level causing a fall, the weir should be erected in this vicinity unless other considerations render another choice desirable.

From the point of view of economy to be obtained in the construction of the installation, the summits of reverse slopes corresponding to sections where the river bed is restricted, would be selected.

There is also need to consider whether it is preferable to place the dam near the up-stream boundary of the property or in the neighbourhood of the down-stream end.

If the dam be built up-stream its height will be small, but a very long and consequently very costly penstock may require to be made.

If, on the other hand, the weir be placed towards the lower end of the property, it will need to be very high and, according to the configuration of the ground as given by the transverse sections, the banks on the up-stream side might be drowned to a considerable extent.

It is necessary therefore to seek to reconcile these different requirements, and to so choose the site for the dam as to reduce the total cost of *weir*, *penstock*, and *tail-race* as much as possible.

One of two arrangements may be adopted, the works may be situated either on the dam or elsewhere.

In the first case only the cost of placing the works in a different profile would have to be considered, and the most economical would be chosen, provided that no unreasonable drowning of the river banks would be caused.

In the second case it would be necessary to bring in the cost of the penstock and tail-race.

It is clear that the latter ought to discharge as far down the stream as possible within the property boundary, if it is desired to use the greatest possible amount of fall.

Let us suppose that the weir is to be constructed at one of the sections near the down-stream end and that the top of the dam is stopped at a height of 28·9 feet, the height at which the water may be kept back during drought. As the water level corresponding to ordinary floods is supposed to be 30·6 feet, a sheet of water 1·7 feet thick will flow over the weir at such times.

As it is necessary to arrange the waste sluice at the dam, the dam is not made the full length of the transverse section of the valley at the level of the crest, but only the length of the water-line during ordinary floods before the erection of the dam. This length being in the chosen numerical example 48·5 feet, this will be the value of the length of the dam.

The sluice gates come between the dam and the banks of the head-race. These banks are built 20 inches above the level during ordinary floods.

The power-house is constructed on this canal to the right of the profile in the direction of the current. Lastly the tail-race is made so as to discharge into the stream at the boundary of the property.

58. Construction of the Dam. We must now consider

the construction of the dam. Its site has been chosen already, and its height is to be such that the water level during drought is to be maintained at 28·9 feet.

The altitude of the bed of the stream at the spot where the dam is to be constructed is 21·8 feet. The foundation will be a concrete bed whose upper surface will be 8 inches below the bed of the stream. Hence the height of the dam will be equal to :

$$h = 28{\cdot}9 - 21{\cdot}8 + {\cdot}7 = 7{\cdot}8 \text{ feet.}$$

It is necessary to consider the pressure exerted on the upstream face of the dam during ordinary floods ; at such times the dam crest is covered with a sheet of water whose thickness is 1·7 feet, so that the depth of the column of water pressing on the dam is increased by this amount.

This pressure F has for its value per foot of length of the dam,

$$F = 62{\cdot}4 \times 7{\cdot}8 \times \left[\frac{7{\cdot}8}{2} + 1{\cdot}7\right] = 2720 \text{ lbs.}$$

The point of application of this force is at a height l from the bottom of the water, where :

$$l = \frac{\frac{7{\cdot}8^2}{3} + 7{\cdot}8 \times 1{\cdot}7}{7{\cdot}8 + 2 \times 1{\cdot}7} = 3 \text{ ft.}$$

The thickness b of the crown of the dam in order that the coefficient of discharge established for a broad crested dam may apply, must be at least equal to one and a half the thickness of the sheet of water passing over the crown.

Let us therefore adopt the value :

$$b = 1{\cdot}5 \times 1{\cdot}7 = 2{\cdot}55,$$

or say 2·5 feet in round numbers.

In order to determine the thickness B of the base of the dam, we will make use of the formula established in books on structures and which we here reproduce :

$$B = \frac{-b\,(p - dh) + \sqrt{pb^2\,(3p - dh) + 6pFl\,.\,\frac{2p - dh}{dh}}}{2p - dh}.$$

In this formula,

$b = 2\cdot5$ feet, the thickness of the dam crest.

$h = 7\cdot8$ feet, the height of the dam.

$F = 2720$ lbs., the normal pressure of the water per foot of length of the dam.

$l = 3$ feet, the height of the point of application of the resultant pressure above the bed.

$d = 144$ lbs., the weight of a cubic foot of masonry.

$p = 8200$ lbs. per sq. ft., the permissible load on masonry.

Using these values in the above formula and calculating out, it is found that:

$$B = 3\cdot72 \text{ feet.}$$

All the dimensions of the dam are now known and the weight P per foot run can be calculated. It will be given by the equation:

$$P = \frac{B + b}{2} \times h \times d = \frac{3\cdot72 + 2\cdot5}{2} \times 7\cdot8 \times 144,$$

from which: $P = 3490$ lbs.

The weight P acts vertically downwards, while the pressure F is exerted in the horizontal direction. Therefore the resultant of these two forces acts in the direction of the diagonal R of the rectangle of which F and P are the sides, and consequently the value of this resultant is:

$$R = \sqrt{P^2 + F^2} = \sqrt{(12\cdot2 + 7\cdot4)\,10^6} = 4420 \text{ lbs.}$$

It is advisable to see if the structure thus calculated is sufficiently stable. To do this the parallelogram of forces is constructed to scale as in the diagram traced in figure 48. It will then be found that the resultant R cuts the base of the dam at a distance of three inches beyond the outer edge.

It is therefore necessary to increase B, let us take as a new value 4·5 feet, it will then be found that P is increased to 3930, and that the resultant falls within the base at a distance *mn* of 6½ inches from the outer edge.

Now the maximum value of the pressure which is exerted on this edge is :

$$p = \frac{2R}{3mn} = 5700 \text{ lbs.}$$

Thus the pressure per square foot near the edge is considerably less than the allowable crushing stress. This result shows that the thickness B derived by the calculation is sufficient and may be accepted.

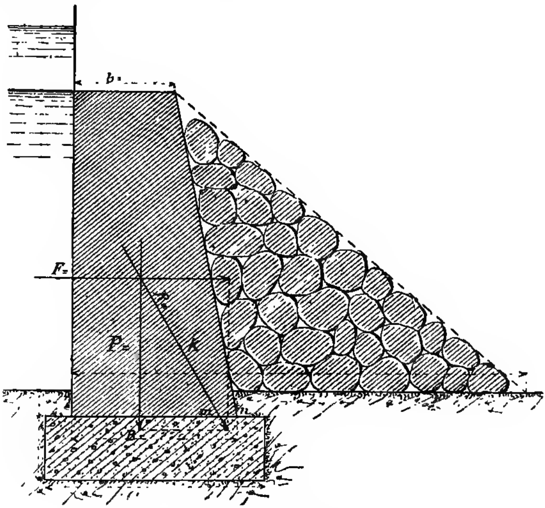

Fig. 48. Dam wall.

The dam is constructed on a concrete foundation 20 inches thick and levelled off at 8 inches below the bed of the stream.

This dam 43·5 feet in length in the example chosen is built into the right bank so that the water cannot find any passage on this side.

On the down-stream side the weir is covered with a sloping heap of stones so as to break the fall of the column of water which would otherwise very quickly undermine the structure and render it insecure. In cases where the water is capable of carrying tree trunks and boulders during floods, it is also

advisable to protect the up-stream face with a similar pile of rock.

On the left bank, the dam is supported by a pier 4 feet thick, whose crown is at an altitude of 36·3 feet and has a breadth of 5 feet 8 inches. This pier serves to support one end of the foot-bridge used in working the gates. These latter are situated in the interval left free between the above-mentioned pier and a platform 22 feet in width lying between the bed of the stream and the head-race.

59. Flow through waste sluice. It is now necessary to estimate the discharge through the waste sluice. This calculation ought to be carried out for the case of ordinary floods. These gates ought in this case to allow that quantity to pass which corresponds to the floods considered, less the amount which passes over the dam. This latter quantity is given by the relation:

$$q = 0{\cdot}385 \times L \times \sqrt{2g \times 1{\cdot}7},$$

in which formula:

$0{\cdot}385$ is the coefficient of contraction of a sheet of water flowing over a thick-walled dam;

$L = 43{\cdot}5$ feet, the length of the dam;

$g = 32$;

$1{\cdot}7$ feet is the depth of water above the level of the crest; hence the flow through the sluice gates per second must be:

$$Q = 1350 - 0{\cdot}385 \times \sqrt{2 \times 32 \times 1{\cdot}7} \times 43{\cdot}5,$$

namely:

$$Q = 1175 \text{ cubic feet.}$$

The sluice gates are so made that their sill is level with the stream bed and so that when fully open their lower edges are as high as the surface of the water.

In order to obtain the width D of the gates, it must be observed that the upper layers of water flowing through them, flow as over a dam, while the lower layers below a depth h' equal to the normal water level at a definite distance down-

stream from the dam, flow in horizontal stream-lines as through a rectangular drowned orifice.

If we denote by:

$h = 7{\cdot}8$ feet, the total depth of water on the up-stream side of the dam;

$h' = 3{\cdot}08$ feet, the depth of water down-stream;

$h'' = h - h' = 4{\cdot}72$ feet, the depth of the sheet of water flowing as over a dam, then we have the relation:

$$Q = m \,.\, D \,.\, h' \times \sqrt{2gh''} + m' \,.\, L \,.\, h'' \times \sqrt{2gh''}.$$

In the first member on the second side of the equation will be recognised the expression for the flow through a drowned orifice under a head h'', and in the second term the expression for the amount of water flowing over a dam will be recognised.

m and m' are contraction coefficients having for their respective values:

$$m = 0{\cdot}62 \text{ and } m' = 0{\cdot}41.$$

Since $L = D$, the above expression may be written:

$$Q = D \times \sqrt{2gh''} \times (m \,.\, h' + m'h''),$$

or: $$D = \frac{1175}{(0{\cdot}62 \times 3{\cdot}08 + 0{\cdot}41 \times 4{\cdot}72) \times \sqrt{2 \times 32 \times 4{\cdot}72}},$$

and finally on replacing the letters by their numerical values:

$$D = 17{\cdot}6 \text{ feet.}$$

60. Construction of the gates. Having been given the quantity of water which must flow through the sluice gates in ordinary floods we have calculated the total width of these gates, namely 17·6 feet.

Obviously this width must be divided into a certain number of equal parts, so as to give practical dimensions to the gates. If four gates are used, each will have a width:

$$l = \frac{17{\cdot}6}{4} = 4{\cdot}4 \text{ feet.}$$

Further, these gates should be so arranged that they can be raised the full height of the dam plus the thickness of the

sheet of water passing over the dam in ordinary floods; this height therefore has the value:

$$h = 7{\cdot}8 + 1{\cdot}7 = 9{\cdot}5 \text{ feet.}$$

Fig. 49 shows details of gate construction for which wood has been used as being cheaper than other materials. There are five upright posts P which form the framing of the gates. At their lower ends these uprights are all fixed to a beam Q_0 which forms the sill of the gates. At their upper ends they are connected by a couple of cross pieces C attached by bolts.

The end posts are embedded in the masonry at each side of the gates; the intermediate uprights are supported by struts A and B, connected together by cross strips L and attached by tenon joints and angle irons to the uprights and to the sleepers T; these latter are placed in the bed of the stream normal to the uprights. The sleepers and the sill beam are mortised and nailed together and are embedded in concrete 8 inches thick forming the radish of the gates.

The gates slide in rebates or grooves cut in the face of the uprights. They are formed of superposed boards, tongued and grooved and nailed to one another. A strip of wood E carries the lifting rack which on one side gears with a pinion and on the other side presses against an iron bar, through an intermediate iron strip attached to the wood to diminish friction. For the same purpose the down-stream edges of the gates are furnished with copper strips which slide over strips of iron riveted in the grooves in the uprights.

The gates are raised by means of a small windlass driven by a winch. On the axle of the latter is mounted a pinion which gears with a toothed wheel fixed on a parallel axle; this latter axle carries a second pinion which is the one in gear with the rack. In this way the force exerted by the operator is multiplied, of course at the expense of speed in raising the gates.

The winch spindle has a ratchet wheel and pawl such as is commonly used with windlasses, for the purpose of holding the gate at the height desired.

The footbridge, shown in transverse section in the figure, runs the whole length of the gates; it is formed of two beams whose ends are carried on masonry; these beams are also supported at intermediate points by brackets attached to the upright posts and held by stays inclined at 45 degrees. The bridge is provided with a handrail throughout its length in order to allow access to the different windlasses in perfect safety.

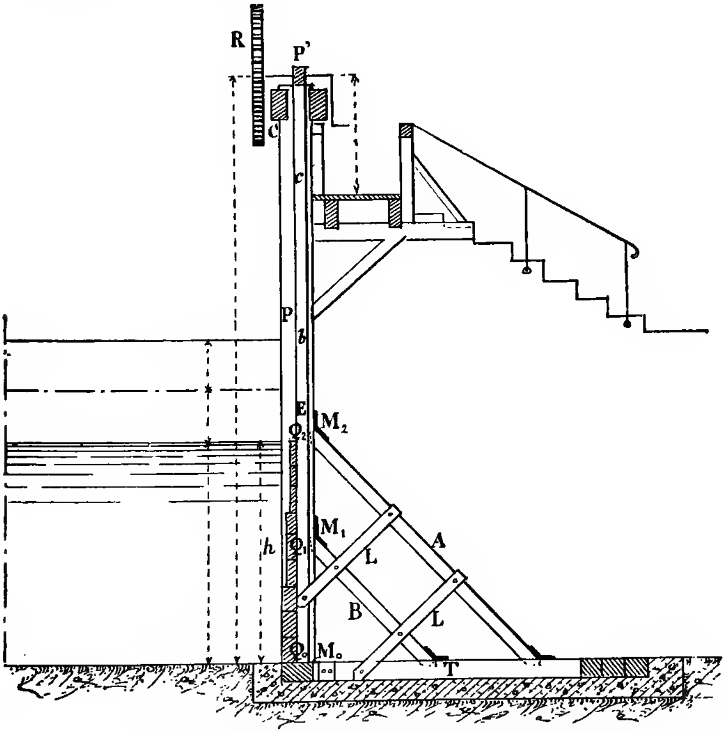

Fig. 49. Details of sluice-gate construction.

61. Calculations of strength. Let us now estimate the necessary section for the posts or uprights of the gate frame.

Evidently each intermediate post bears the same pressure as does each gate as a consequence of the water on the up-stream side. If therefore we use l to represent the width of the gate, h its height, the hydraulic pressure may be expressed by:

$$P = 62{\cdot}4l \times h \times \frac{h}{2} = 31{\cdot}2 \times l \,.\, h^2.$$

Now this pressure transmitted to the upright gives rise to three normal reactions Q_0, Q_1 and Q_2 which act respectively at the three points of support M_0, M_1 and M_2, such that:

$$Q_0 + Q_1 + Q_2 = P.$$

Therefore the strength and necessary section of the piece M_0M_2 of the upright subject to the pressure of the water can be calculated, by regarding it as a beam supported at three points and loaded according to the law of hydrostatic pressure.

The calculation of the strength of materials requires knowledge of the value of the maximum bending moment on the piece considered. It may be shown by a rather long calculation that in the case under consideration, this maximum bending moment occurs at the point M_1, therefore it is the bending moment at this point, and which we will denote by m_1, which must be considered.

To do this, we must take account of all the forces on one side of M_1, say below. These forces are the reaction Q_0 at the support M_0 and hydrostatic pressure distributed between M_0 and M_1.

The moment due to the first force is evidently equal to:

$$m' = Q_0 \times \frac{h}{2}.$$

That due to the second force may be shown to have the value:

$$m'' = 62{\cdot}4 \times \frac{5 \times lh^3}{48}.$$

Hence the bending moment on the section M_1 may be expressed by:

$$m_1 = Q_0 \times \frac{h}{2} - \frac{62{\cdot}4 \times 5 \times lh^3}{48},$$

since the second moment must be subtracted from the first as the two forces causing them act in contrary directions.

To determine the value of m_1 from this, the reaction Q_0 at the point M_0 must be known. It is therefore necessary to establish a second relation, which we shall find on investigating the value of the bending moment at M_2.

This moment m_2 has three component parts; the moment m_1 at the point M_1, the moment $Q_1 \times \frac{h}{2}$, which is the result of the reaction Q_1 at the same point, and that due to the hydrostatic pressure distributed from M_1 to M_2. This latter moment has for its value:

$$n = \frac{62{\cdot}4 \times lh^3}{24};$$

hence the moment m_2 is given by the expression:

$$m_2 = m_1 + Q_1 \times \frac{h}{2} - \frac{62{\cdot}4 \times lh^3}{24}.$$

Now m_2 is simply the sum of the moments acting below the point M_2, and this sum must be zero since the forces acting above M_2 and which are in equilibrium with those below are themselves zero. Therefore:

$$m_2 = m_1 + Q_1 \times \frac{h}{2} - \frac{62{\cdot}4 \times lh^3}{24} = 0,$$

from which may be deduced:

$$m_1 = \frac{62{\cdot}4 \times lh^3}{24} - \frac{Q_1 \times h}{2}.$$

In this way a second value is obtained for m_1 in terms of a new unknown Q_1. On combining these two results we get the relation:

$$Q_0 + Q_1 = \frac{7 \times 62{\cdot}4 \, . \, lh^2}{24},$$

which may be added to that established in the first place:

$$Q_0 + Q_1 + Q_2 = P.$$

Finally, we have four unknowns and only three equations

connecting them; a fourth relation connecting the reactions at the supports is therefore needed.

This last is obtained by having recourse to formulae, giving at every point the inclination of the middle layer of the post. By this means, after a somewhat laborious calculation, the following result is arrived at:

$$Q_0 - Q_1 = \frac{9 \cdot 35}{48} lh^2.$$

On connecting this last equation with the preceding ones, the values of the reactions at the points of support are easily obtained:

$$Q_0 = \frac{441 \cdot 5}{48} . lh^2,$$

$$Q_1 = \frac{432 \cdot 1}{48} . lh^2,$$

$$Q_2 = \frac{312}{24} . lh^2.$$

It now suffices to use these values of Q_0 or Q_1 in the equations for m_1 to obtain the value of this quantity:

$$m_1 = \left(\frac{62 \cdot 4}{24} - \frac{432 \cdot 1}{96}\right) lh^3 = -\frac{182 \cdot 5}{96} lh^3.$$

The value of m_1 being known, the section of the post may be deduced by the general formula:

$$m_1 = R \times \frac{I}{v},$$

where I is the moment of inertia of the section about its axis, v is the distance of the extreme outer layer from the neutral axis, and R is the allowable safe stress.

Using a and b to represent the dimensions of the section of the post supposed rectangular, b being the dimension in the direction parallel to the action of the forces, then:

$$\frac{I}{v} = \frac{ab^2}{6};$$

hence: $$m_1 = \frac{R \, . \, a \, . \, b^2}{6};$$

and finally: $$b = \sqrt{\frac{6m_1}{Ra}}.$$

It is necessary to replace m_1 in this equation by its value deduced from the formula established above, a suitable value may then be given to a. Let us take in this actual case:

$$a = 10 \text{ inches},$$

and: $$R = 500 \text{ lbs. per square inch.}$$

These different values put in the formula for b enable us to calculate this second dimension of the post's section.

We have now to calculate the dimensions of the struts A and B which abut on to the uprights (fig. 48). These pieces must withstand forces Q_2 and Q_1, which are exerted at the points M_2 and M_1, and which tend to bend and compress them. It is advisable to calculate these struts as supports subject to longitudinal forces.

Using F_1 and F_2 to represent these longitudinal forces respectively, then evidently:

$$F_2 = Q_2 \times \sqrt{2},$$
$$F_1 = Q_1 \times \sqrt{2},$$

if the struts are assumed to be inclined at 45 degrees; for F_1 and F_2 will be hypotenuses of triangles of force such as M_1M_0T, whose sides such as M_0T represent the forces Q_0 and Q_1.

The formula connecting the dimensions of a strut of length l, and of square section of side c inches, subject to a longitudinal force N, is:

$$c = \sqrt[4]{\frac{N \times l^2}{1290}}.$$

But as these struts are exposed to the action of air and water, it is necessary to halve the stress which may be applied to them, or, what comes to the same thing, to double the value of N. Now, in this particular case, we have:

for the piece A: $N_2 = 2Q_2 \sqrt{2}, \quad l_2 = h \sqrt{2};$

and for the piece B:

$$N_1 = 2Q_1\sqrt{2}, \quad l_1 = \frac{h}{2}\sqrt{2}.$$

Using these values in the expression for c, we have:

$$c_1 = \sqrt[4]{\frac{Q_1\sqrt{2} \times 7{\cdot}8^2}{1290}},$$

$$c_2 = \sqrt[4]{\frac{2Q_2\sqrt{2} \times 2 \times 7{\cdot}8^2}{1290}}.$$

In order to work out these expressions it is necessary to know Q_1 and Q_2; the values of these are readily obtained on working out the expressions established above, to wit:

$$Q_1 = \frac{432{\cdot}1}{48}lh^2 = \frac{432{\cdot}1 \times 4{\cdot}4 \times 7{\cdot}8^2}{48} = 2410,$$

and: $$Q_2 = \frac{312}{24}lh^2 = \frac{312 \times 4{\cdot}4 \times 7{\cdot}8^2}{24} = 3480.$$

Replacing Q_1 and Q_2 by these values in the foregoing expressions for c, effecting the calculations and extracting the fourth root, or, what comes to the same thing, working out the square root twice, it is found that:

$$c_2 = 5{\cdot}5 \text{ inches},$$

and: $$c_1 = 3{\cdot}6 \text{ inches}.$$

In practice the two pieces would each be made the same in section, and the greater dimension would be chosen, that is to say:

$$c = 5{\cdot}5 \text{ inches}.$$

We will calculate also the dimension b of the uprights. We found that the expression for the bending moment was:

$$m_1 = -\frac{182{\cdot}5}{96}.lh^3.$$

Replacing the letters by their numerical values, we get:

$$m_1 = -\frac{182{\cdot}5 \times 4{\cdot}4 \times 7{\cdot}8^3}{96}$$

$$= -3970 \text{ lb.-feet, or } -47{,}640 \text{ lb.-inches}.$$

This value put in the equation for b already established gives:

$$b = \sqrt{\frac{6m}{R \cdot a}} = \sqrt{\frac{6 \times 47{,}640}{500 \times 10}} = \sqrt{57 \cdot 2} = 7 \cdot 6 \text{ inches,}$$

say $b = 8$ inches.

The thickness of the boards forming the gates still remains to be calculated. As is seen in fig. 49, the gates are made in three portions, the thickness of the lowest portion being the greatest. Therefore each part should be calculated to bear the maximum stress it will have to support, which obviously is exerted on the lowest plank of the portion.

Consider a horizontal strip 1 inch in depth at the lower edge of the upper portion, at a depth of $\frac{7 \cdot 8}{3} = 2 \cdot 6$ feet below the level of the water. The pressure exerted per foot of length on the centre of the strip under consideration, at a depth of 2·6 feet has in consequence the value:

$$P = 62 \cdot 4 \times 2 \cdot 6 \times \tfrac{1}{12} = 13 \cdot 1 \text{ lbs.}$$

The piece under consideration is a rectangular section prism of base 1 inch and thickness h unknown. It may be regarded as a loaded beam resting on two horizontal supports 4·4 feet apart, the load P being uniformly distributed.

In this case the bending moment is a maximum at the middle of the length l and its value is:

$$m = \tfrac{1}{8} P \cdot l^2 = \tfrac{1}{8} \cdot 13 \cdot 1 \, (4 \cdot 4)^2 = 31 \cdot 7 \text{ lb.-ft. or } 380 \cdot 4 \text{ lb.-inches.}$$

Making use of the same formula as before:

$$m = R \times \frac{I}{v},$$

in which:

$$\frac{I}{v} = \frac{bh^2}{6},$$

from which:

$$m = R \times \frac{bh^2}{6},$$

and finally:

$$h = \sqrt{\frac{6m}{R \cdot b}} = \sqrt{\frac{6 \times 380 \cdot 4}{500 \times 1}} = 2 \cdot 14 \text{ inches.}$$

Similarly for the second portion of the gate considering the lower strip situated at a depth of 5·2 feet below the level of the water:

$$P = 62{\cdot}4 \times 5{\cdot}2 \times \tfrac{1}{12} = 26{\cdot}2 \text{ lbs.},$$

and the maximum bending moment is:

$$m = \tfrac{1}{8} . Pl^2 = \tfrac{1}{8} \times 26{\cdot}2 \times (4{\cdot}4)^2 = 63{\cdot}4 \text{ lb.-ft.},$$

or:

$$m = 760{\cdot}8 \text{ lb.-inches},$$

and consequently:

$$h = \sqrt{\frac{6m}{R . b}} = \sqrt{\frac{6 \times 760{\cdot}8}{500 \times 1}} = 3{\cdot}02 \text{ inches.}$$

For the third piece, placed at the bottom of the gate at the greatest depth of 7·8 feet the pressure on the 1 inch strip will be:

$$P = 39{\cdot}3 \text{ lbs.}$$

For the maximum bending moment as before:

$$m = 1141{\cdot}2 \text{ lb.-inches},$$

from which may be deduced:

$$h = 4{\cdot}27 \text{ inches.}$$

The thickness of the boards of which the three parts of the gate are composed are thus determined.

It is important to calculate the weight of the gate as a preliminary to the erection of the lifting mechanism. If we assume that the boards are of oak, the weight per cubic foot is about 57·5 lbs., but we will allow 62·5 lbs. as taking into account the metal work and rack attached to it.

The weight of each gate will then be calculated as follows:

$$62{\cdot}5 \times 4{\cdot}4 \times 2{\cdot}6 \times \tfrac{1}{12} (2{\cdot}14 + 3{\cdot}02 + 4{\cdot}27) = 561 \text{ lbs.}$$

Or say 600 lbs. in round numbers.

For the erection of the gate mechanism, it is necessary to bear in mind that the force to be overcome in raising each gate is made up of: (1) the weight of the gate; (2) the friction in the grooves in the posts, the value of which is reduced as much as possible by covering the bearing surfaces with metal; (3) the friction of the teeth in the gearing.

The force due to the pressure of the water on the gates has the value:

$$f \times F = 0{\cdot}2 \times 8330 = 1666 \text{ lbs.}$$

In this formula, f is the coefficient of friction of the metallic surfaces under water; we have assumed this to be equal to 0·2; F is the total pressure on each gate; which is equal to:

$$F = \frac{62{\cdot}4 \times 4{\cdot}4 \times 7{\cdot}8^2}{2} = 8330 \text{ lbs.}$$

It follows that the force R to be applied to the rack is given by the sum:

$$R = P + F = 600 + 1666,$$

say 2300 lbs. in order to take into account additional resistances caused by foreign matter getting between the gate and the grooves.

If N be used to denote the number of teeth in the pinion gearing with the rack, the formula giving the value of the force to be exerted on the said pinion is:

$$R' = R\left(1 + f \times \frac{\pi}{N}\right).$$

Supposing that the pinion has ten teeth, and that the coefficient of friction is 0·15, we have:

$$R' = 2300\left(1 + \frac{0{\cdot}15 \times 3{\cdot}14}{10}\right) = 2400 \text{ lbs.}$$

By formulae in mechanics whose application is outside the range of this work, it is found that the diameter of the pinion should be equal to 7 inches.

It will be seen that in order that the effort to be put forth by the workman at the handwheel should not exceed 60 lbs. it is necessary that the gear train should be such as to multiply his effort by 40, that is to say, between the effort put forth and the resistance to be overcome there is the relation:

$$\frac{2400}{60} = 40.$$

The toothed-wheel mounted on the pinion spindle ought

to have a diameter equal to 3·12 feet and the pinion mounted on the same spindle as the handle should have its diameter reduced to 4·05 inches.

The winch is $2\frac{1}{2}$ feet in length.

To complete the problem, only the dimensions of the lintel of the framing need be calculated. This is composed of the two horizontal beams C connecting the posts. It is on these that the pulls are exerted in working the gates.

The maximum bending moment will occur at the middle of the length of the two beams, and as each has to support only half the force, this moment will be:

$$m = \tfrac{1}{8}P \,.\, l = \frac{1200 \times 4{\cdot}4}{8} = 660 \text{ lb. ft.},$$

considering the pieces as beams fixed at the ends.

Let b and h be the dimensions of the section of either piece, then

$$\frac{I}{v} = \frac{1}{6} \,.\, bh^2,$$

and if we make: $$b = \frac{h}{2},$$

this becomes finally: $$\frac{I}{v} = \frac{1}{12} \,.\, h^3,$$

whence: $$h = \sqrt[3]{\frac{12m}{R}} = \sqrt[3]{\frac{12 \times 12 \times 660}{500}}$$

$$= 5\tfrac{3}{4} \text{ inches, or say 6 inches.}$$

The second dimension will therefore be:

$$b = \frac{h}{2} = 3 \text{ inches.}$$

62. Estimation of height of available fall. The head-race starts at profile No. 6 and reaches the power-house at section No. 8; the dam is built at profile No. 7 between the two former, as we have seen. The construction of this canal requires, on the left bank, a retaining wall which has to withstand the thrust of the earth, and on the right bank

a wall of earth which has principally to resist the pressure of the water.

The retaining wall on the left bank leaves the side of the river at profile No. 6; the pier has its origin to the right of the dam and forms one of the points of support for the gates.

The bottom of the head-race in this section is at the same height as the river bed in this profile. It is advisable in all cases that the canal bed should not be below the river bed, because it would then act as a spill-basin and collect sand and stones carried along by floods.

The retaining walls and the paved radish rest on a bed of concrete whose thickness varies from 26·9 feet under the radish to 1·6 feet under the walls.

At profile No. 7 the level of the water is raised to a height of 28·9 feet during low water, and to 30·6 feet during ordinary floods after the erection of the dam.

The crest of the wall on the left bank of the head-race must therefore be as high as this on the right of profile No. 7; but it is advisable to make it 20 inches higher so as to run no risk of it being drowned during exceptional floods. Obviously the path of this wall ought to follow fairly closely the contour line corresponding to the water line during ordinary floods. Therefore this wall will be made to follow a path lying on the contour plan between the lines of level of height 30 and 33 feet.

Let us adopt for the slope of the canal, on leaving section No. 6 an inclination of 0·0002 ft. per foot, which for the measured length of 180 feet on the plan, represents a loss of available fall of 0·036 ft. between the commencement of the canal and the power-house.

The tail-race reaches from the installation to profile No. 11, which is the property boundary. The construction of this canal is like that of the head-race. On giving the bed a slope of 0·0004 ft. per foot, the loss of fall in the 394 feet which is the length of the canal will be 0·158 ft.

In order to calculate the height of useful fall, it must be recalled that between the extreme limits of the property,

extending from profile No. 1 to profile No. 11 the total difference in level was during drought:

$$28{\cdot}9 - 19{\cdot}0 = 9{\cdot}9 \text{ ft.},$$

and during ordinary floods:

$$30{\cdot}6 - 21{\cdot}2 = 9{\cdot}4 \text{ ft.}$$

It follows that the useful fall during drought is:

$$H = 28{\cdot}9 - 19{\cdot}0 - 0{\cdot}036 - 0{\cdot}158 = 9{\cdot}7 \text{ ft.}$$

The quantity flowing per second during drought being 664 cu. ft. the power of the fall will then be:

$$P = 664 \times 62{\cdot}4 \times 9{\cdot}7 = 402{,}000 \text{ ft.-lbs.},$$

or in horse-power:

$$P = \frac{402000}{550} = 730 \text{ horse-power.}$$

In the case of ordinary floods, the available fall will be:

$$H_1 = 30{\cdot}6 - 21{\cdot}2 - 0{\cdot}036 - 0{\cdot}158 = 9{\cdot}2 \text{ ft.}$$

The available fall is therefore less during ordinary floods; if it is desired to maintain the same power, the flow through the hydraulic machinery needs to be increased; evidently the flow ought to be:

$$Q = \frac{9{\cdot}7}{9{\cdot}2} \times 664 = 700 \text{ cu. ft.}$$

As the gates can deliver 1175 cubic feet, until the flow in the river exceeds this, the level of the water will not pass the crest of the dam, namely the height 28·9 feet; nevertheless the water level in the tail-race will rise, and if we consider the average level, assuming that the tail-race level is midway between that for low water and for ordinary floods, say at the height 20·1 feet, then the fall under these mean conditions would be:

$$H_2 = 28{\cdot}9 - 20{\cdot}1 - 0{\cdot}036 - 0{\cdot}158 = 8{\cdot}6 \text{ ft.}$$

With such a reduced fall, the flow through the turbines to develop the same power would need to be:

$$Q = \frac{9{\cdot}7}{8{\cdot}6} \times 664 = 750 \text{ cu. ft.}$$

The flow is therefore a maximum, and it is for this volume of water that the sections of the head and tail-races should be estimated.

63. Calculation of canal sections. For the supply canal, we will make use of the formula indicated earlier (§ 18):

$$\frac{S}{K} \times i = b_1 \times u^2,$$

in which formula S is the area of transverse section of the flowing water, K is the wet perimeter and i is the slope of the canal. The velocity u is connected with the section S and the delivery Q by the relation:

$$Q = S \times u,$$

from which:

$$S = \frac{Q}{u}.$$

Adopting in this case the following values:

$$\begin{aligned} i &= 0{\cdot}0002 \text{ ft. per ft.} \\ b_1 &= 0{\cdot}00012 \\ Q &= 750 \text{ cu. ft.} \\ u &= 2{\cdot}8 \text{ ft. per sec.} \end{aligned}$$

we may deduce:

$$S = \frac{Q}{u} = \frac{750}{2{\cdot}8} = 268 \text{ sq. ft.}$$

Now the depth of water in profile No. 7 is 7·15 feet, which enables us to obtain the width of the canal:

$$l = \frac{268}{7{\cdot}15} = 37{\cdot}5 \text{ ft.}$$

But it is necessary to verify whether the velocity and the slope assumed *a priori* are compatible with one another. For this purpose we deduce from the first relation:

$$i = b_1 \times \frac{K}{S} \times u^2.$$

If now we put in the second member of this equation the

numerical values represented there as calculated or assumed, we obtain:

$$i = 0{\cdot}00012 \times \frac{51{\cdot}8}{268} \times (2{\cdot}8)^2 = 0{\cdot}00018.$$

Hence it would suffice if the slope were 0·00018 ft. per foot to deliver the required quantity through the canal at a speed of 2·8 feet per second, the slope of 0·0002 ft. per foot is therefore rather more than sufficient, but is so near that obtained by direct calculation that it may be regarded as quite suitable and adhered to.

Ordinarily the tail-race would be made the same width as the head-race; the data for this canal will therefore be:

$$i = 0{\cdot}0004.$$

$$b_1 = 0{\cdot}00012.$$

$$Q = 750 \text{ cu. ft.}$$

The depth and mean velocity of the water remain to be determined. From the above relation we may write:

$$u^2 = \frac{S}{K} \times \frac{i}{b_1};$$

and since:

$$u = \frac{Q}{S}.$$

and:

$$u^2 = \frac{Q^2}{S^2},$$

we may write:

$$\frac{Q^2}{S^2} = \frac{S}{K} \times \frac{i}{b_1},$$

from which may be deduced:

$$\frac{S^3}{K} = Q^2 \times \frac{b_1}{i}.$$

Moreover: $S = l \times h = 37{\cdot}5 \times h,$

and also: $K = l + 2h = 37{\cdot}5 + 2h.$

Substituting these values in the above equation, it becomes:

$$\frac{37{\cdot}5^3 \times h^3}{37{\cdot}5 + 2h} = 750^2 \times \frac{0{\cdot}00012}{0{\cdot}0004}.$$

Finally, simplifying and working out the calculations:

$$55{,}500h^3 = 337{,}500h + 6{,}330{,}000.$$

This may now be put in the form:

$$h^2 = 6\cdot08 + \frac{114}{h},$$

and solved by successive approximations. It will be found that finally:

$$h = 5\cdot3.$$

The mean speed of the water in the tail-race under average conditions may be deduced as follows:

$$u = \frac{750}{37\cdot5 \times 5\cdot3} = 3\cdot8 \text{ ft. per sec.};$$

finally, checking the value of i, we find:

$$i = b \times \frac{K}{S} \times u^2 = \frac{0\cdot00012 \times 48\cdot1}{37\cdot5 \times 5\cdot3} \times 3\cdot8^2 = 0\cdot00042.$$

64. Calculation of Retaining Walls. The section of the retaining walls forming the canal banks might be calculated by the simplified method due to Poncelet, which is explained in books on stability and strength.

It would be found that under these conditions the thrust of the earth per foot of length of retaining wall 11·1 feet in height on the left bank of the head-race would be 2410 lbs.; the point of application of this force is at one-third of the height of the inner face of the wall from the base. Suitable formulae give for the base, B = 4 feet, and for the width of the crest, 2·6 feet, assuming an inner facing with a batter of $\frac{1}{8}$.

The resultant of the pressure F and the weight P of the wall which is equal to 5300 lbs. per foot of length, is given by the equation:

$$R = \sqrt{P^2 + F^2 + 2P \times F \times \frac{1}{\sqrt{1 + i^2}}},$$

where i is the natural slope of the earth to be retained; for an inclination of 50 degrees, $i = 1\cdot19$.

Hence:

$$R = \sqrt{5300^2 + 2410^2 + 2 \times 5300 \times 2410 \times \frac{1}{\sqrt{2 \cdot 42}}} = 7100 \text{ lbs.}$$

On tracing the Poncelet diagram and measuring on the drawing paper the length d comprised between the outer edge of the wall and the point where the base is cut by the resultant, it is found to be 7 inches. Consequently, the fundamental formula, relating to the maximum pressure at the outer edge, gives:

$$p = \frac{2R}{3 \times d} = \frac{2 \times 7100}{3 \times \frac{7}{12}} = 8110 \text{ lbs. per square foot,}$$

which is less than the practical allowable crushing stress, and is quite safe; the base thickness of 4 feet is therefore more than sufficient.

The wall of the head-race, although buttressed by the earth, is calculated as though subject to the full pressure of the water on the inside and quite unsupported on the outside. Under these conditions it is found that the base should be 3·44 feet in width.

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